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A Study on Mini-Max Theorem and Its Applications

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Abstract:

To present a survey of existing mini max theorems, To give applications to elliptic differential equations in bounded domains, To consider the dual variational method for problems with continuous and discontinuous nonlinearities, To present some elements of critical point theory for locally Lipschitz functional and give applications to fourth-order differential equations with discontinuous nonlinearities, To study homoclinic solutions of differential equations via the variational methods.

Keywords:

Discontinuous nonlinearities, Lipschitz functional, minimax theorems.

Introduction:

In linear algebra and functional analysis, the min-max theorem, or variational theorem, or Courant–Fischer–Weyl min-max principle, is a result that gives a variational characterization of eigenvalues of compact Hermitian operators on Hilbert spaces. It can be viewed as the starting point of many results of similar nature. This article first discusses the finite-dimensional case and its applications before considering compact operators on infinite-dimensional Hilbert spaces. We will see that for compact operators, the proof of the main theorem uses essentially the same idea from the finite-dimensional argument. In the case that the operator is non-Hermitian, the theorem provides an equivalent characterization of the associated singular values.

The min-max theorem can be extended to self-adjoint operators that are bounded below.

Min-max Theorem

Let A be a n × n Hermitian matrix with eigenvalues $\lambda_1 \leq ... \leq \lambda_k \leq ... \leq \lambda_n$ then

$$\lambda_k = \min\{\max\{R_A(x) \mid x \in U \text{ and } x \neq 0\} \mid \dim(U) = k\}$$

And

$$\lambda_k = \max\{\min\{R_A(x) \mid x \in U \text{ and } x \neq 0\} \mid \dim(U) = n - k + 1\}$$

in particular,

$$\lambda_1 \leq R_A(x) \leq \lambda_n \quad \forall x \in \mathbf{C}^n \backslash \{0\}$$

and these bounds are attained when x is an eigenvector of the appropriate eigenvalues.Also note that the simpler formulation for the maximal eigenvalue λ_n is given by:

$$\lambda_n = \max\{R_A(x) : x \neq 0\}.$$

Similarly, the minimal eigenvalue λ_1 is given by:

$$\lambda_1 = \min\{R_A(x) : x \neq 0\}.$$

Proof

Since the matrix A is Hermitian it is diagonalizable and we can choose an orthonormal basis of eigenvectors $\{u_1, ..., u_n\}$ that is, u_i is an eigenvector for the eigenvalue λ_i and such that $(u_i, u_i) = 1$ and $(u_i, u_j) = 0$ for all $i \neq j$.

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If U is a subspace of dimension k then its intersection with the subspace span{u_k, ..., u_n} isn't zero (by simply checking dimensions) and hence there exists a vector v ≠ 0 in this intersection that we can write as

$$v = \sum_{i=k}^{n} \alpha_i u_i$$

and whose Rayleigh quotient is

$$R_A(v) = \frac{\sum_{i=k}^{n} \lambda_i \alpha_i^2}{\sum_{i=k}^{n} \alpha_i^2} \geq \lambda_k$$

(as all $\lambda_i \geq \lambda_k$ for i=k,..,n) and hence

$$\max\{R_A(x) \mid x \in U\} \geq \lambda_k$$

Since this is true for all U, we can conclude that

$$\min\{\max\{R_A(x) \mid x \in U \text{ and } x \neq 0\} \mid \dim(U) = k\} \geq \lambda_k$$

This is one inequality. To establish the other inequality, chose the specific k-dimensional space V = span{u_1, ..., u_k} , for which

$$\max\{R_A(x) \mid x \in V \text{ and } x \neq 0\} \leq \lambda_k$$

because λ_k is the largest eigenvalue in V. Therefore, also

$$\min\{\max\{R_A(x) \mid x \in U \text{ and } x \neq 0\} \mid \dim(U) = k\} \leq \lambda_k$$

In the case where U is a subspace of dimension n-k+1, we proceed in a similar fashion: Consider the subspace of dimension k, span{u_1, ..., u_k}. Its intersection with the subspace U isn't zero (by simply checking dimensions) and hence there exists a vector v in this intersection that we can write as

$$v = \sum_{i=1}^{k} \alpha_i u_i$$

and whose Rayleigh quotient is

$$R_A(v) = \frac{\sum_{i=1}^{k} \lambda_i \alpha_i^2}{\sum_{i=1}^{k} \alpha_i^2} \leq \lambda_k$$

and hence

$$\min\{R_A(x) \mid x \in U\} \leq \lambda_k$$

Since this is true for all U, we can conclude that

$$\max\{\min\{R_A(x) \mid x \in U \text{ and } x \neq 0\} \mid \dim(U) = n - k + 1\} \leq \lambda_k$$

Again, this is one part of the equation. To get the other inequality, note again that the eigenvector u of λ_k is contained in U = span{u_k, ..., u_n} so that we can conclude the equality.

Counterexample in the non-Hermitian case

Let N be the nilpotent matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Define the Rayleigh quotient $R_N(x)$ exactly as above in the Hermitian case. Then it is easy to see that the only eigenvalue of N is zero, while the maximum value of the Rayleigh ratio is 1/2 That is, the maximum value of the Rayleigh quotient is larger the maximum eigenvalue.

Min-max principle for singular values

The singular values {σ_k} of a square matrix M are the square roots of eigenvalues of M*M (equivalently MM*). An immediate consequence of the first equality from min-max theorem is

$$\sigma_k^{\uparrow} = \min_{S:\dim(S)=k} \max_{x \in S, \|x\|=1} (M^* M x, x)^{\frac{1}{2}} = \min_{S:\dim(S)=k} \max_{x \in S, \|x\|=1} \|Mx\|.$$

Similarly,

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$$\sigma_k^{\downarrow} = \max_{S:\dim(S)=n-k+1} \min_{x\in S, \|x\|=1} \|Mx\|.$$

Cauchy interlacing theorem

Let A be a symmetric n × n matrix. The m × m matrix B, where m ≤ n, is called a **compression** of A if there exists an orthogonal projection P onto a subspace of dimension m such that P*AP = B. The Cauchy interlacing theorem states:

Theorem. If the eigenvalues of A are $\alpha_1 \le ... \le \alpha_n$, and those of B are $\beta_1 \le ... \le \beta_j \le ... \le \beta_m$, then for all j < m + 1,

$$\alpha_j \le \beta_j \le \alpha_{n-m+j}.$$

This can be proven using the min-max principle. Let β_i have corresponding eigenvector b_i and S_j be the j dimensional subspace S_j = span{b_1, ..., b_j}, then

$$\beta_j = \max_{x\in S_j, \|x\|=1}(Bx,x) = \max_{x\in S_j, \|x\|=1}(P^*APx,x) \ge \min_{S_j}\max_{x\in S_j, \|x\|=1}(Ax,x) = \alpha_j.$$

According to first part of min-max, $\alpha_j \le \beta_j$. On the other hand, if we define S_{m-j+1} = span{b_j, ..., b_m}, then

$$\beta_j = \min_{x\in S_{m-j+1}, \|x\|=1}(Bx,x) = \min_{x\in S_{m-j+1}, \|x\|=1}(P^*APx,x) = \min_{x\in S_{m-j+1}, \|x\|=1}(Ax,x) \le \alpha_{n-m+j},$$

where the last inequality is given by the second part of min-max.

Notice that, when n − m = 1, we have $\alpha_j \le \beta_j \le \alpha_{j+1}$, hence the name interlacing theorem.

The von Neumann minimax theorem will be introduced, but the main focus will be on its application, rather than rigorous proof. Some general properties of the expected payoff function will be described along with selective propositions which follow from the minimax theorem. We will then show how these ideas can be applied to find mixed strategy solutions and the value of mx2 and 2xn matrix games.

	y	$1-y$	row min
x	1	3	1
$1-x$	4	2	2
col max	4	3	

The game has no saddle point, so we assume the game to be played with mixed strategies:

$$E(x,y) = xy*1 + 3x(1-y) - (1-x)4y + 2(1-x)(1-y) = -4xy + x + 2y + 2 = -4(x-\frac{1}{2})(y-\frac{1}{4}) + \frac{5}{2}$$

From this expression, it is clear that player 1 can ensure his expectation is at least 5/2 (by choosing x=0.5). And he cannot be sure of more than 5/2 for by taking y=1/4, player 2 can ensure that player 1's expectation id exactly 5/2. Thus player 1 may as well settle for 5/2 and play x=1/2, so as to gain this amount. Similarly, player 2 may as well reconcile himself to get -5/2 and play y=1/4 so as to gain it.

Now we observe that

$$E(x,0.25) \le E(0.5,0.25) \le E(0.5,y)$$

and so (0.5,0.25) is a saddle point of the game. The value of the game is 5/2. For the game with payoff matrix A=(a_{ij}) where i=1,…,m and j=1,…,n, the expected payoff function E is defined on S_mxS_n by

$$E(\tilde{x},\tilde{y}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j$$

where $\tilde{x} = (x_1, x_2, \cdots, x_m) \in S_m$ and $\tilde{y} = (y_1, y_2, \cdots, y_n) \in S_n$.

It turns out (proof omitted) that

$$\max_{x\in S_m}\min_{y\in S_n} E(\tilde{x},\tilde{y}) = \min_{y\in S_m}\max_{x\in S_m} E(\tilde{x},\tilde{y})$$

for every matrix game and so there always exist optimal mixed strategies. This is the content of the famous **von Neumann minimax theorem**.

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From this it will follow that there exist

$\tilde{x}^* \in S_m, \tilde{y}^* \in S_n$ such that

$$E(\tilde{x}, \tilde{y}^*) \le E(\tilde{x}^*, \tilde{y}^*) \le E(\tilde{x}^*, \tilde{y}) \qquad \forall \tilde{x} \in S_m, \tilde{y} \in S_n$$

Consider the matrix game with payoff matrix (a_{ij}) of order mxn. Sets of mixed strategies for players 1,2 are respectively S_m, S_n where

$$S_k = \left\{ (z_1, z_2, \cdots, z_k \in \Re^k : z_i \ge 0, \sum_{i=1}^{k} z_i = 1 \right\}$$

A mixed strategy for a player having at least two non zero components is called a proper mixed strategy.The expected payoff function is the real valued function defined on SmxSn by

$$E(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} x_i y_j$$

The object of player 1 is to choose $\bar{x}$ to maximize $E(x,y)$. The object of player 2 is to choose $\bar{y}$ to minimize $E(x,y)$. If player 1 chooses $x \in S_m$, the payoff to player 1 will be at least $\min_{y \in S_n} E(x,y)$, hence player 1 can expect a payoff of at least $\max_{x \in S_m} \min_{y \in S_n} E(x,y)$. If player 2 uses $y \in S_n$ then the expected payoff to player 1 is at most $\max_{x \in S_m} E(x,y)$ and so player 2 can ensure the expected payoff to player 1 is at most $\min_{y \in S_n} \max_{x \in S_m} E(x,y)$. The principal aim is to determine if $\min_{y \in S_n} \max_{x \in S_m} E(x,y) = \max_{x \in S_m} \min_{y \in S_n} E(x,y)$.

Let i be a pure strategy for player 1 and let $y \in S_n$ be a mixed strategy for player 2. Then

$$E(i,y) = \sum_{j=1}^{n} a_{ij} y_j$$

Now let $x \in S_m$ be a mixed strategy for player 1. Then

$$E(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} x_i y_j = \sum_{i=1}^{m} x_i \sum_{j=1}^{n} a_{ij} y_j = \sum_{i=1}^{m} x_i E(i,y)$$

Similarly, let j be a pure strategy for player 2 and let $x \in S_m$ be a mixed strategy for player 1. Then

$$E(x,j) = \sum_{i=1}^{m} a_{ij} x_i$$

Now let $y \in S_n$ be a mixed strategy for player 2, then

$$E(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} x_i y_j = \sum_{j=1}^{n} y_j \sum_{i=1}^{m} a_{ij} x_i = \sum_{j=1}^{n} y_j E(x,j)$$

Reference:

1. Simons, S. (1998). Minimax and monotonicity. Springer.
2. Sion, M. (1958). On general minimax theorems. Pacific J. Math, 8(1), 171-176.
3. Fan, K. (1953). Minimax theorems. Proceedings of the National Academy of Sciences, 39(1), 42-47.
4. Lin, L. J., & Du, W. S. (2006). Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces. Journal of Mathematical Analysis and Applications, 323(1), 360-370.
5. Komiya, H. (1988). Elementary proof for Sion's minimax theorem. Kodai mathematical journal, 11(1), 5-7.
6. Fan, K. (1952). Fixed-point and minimax theorems in locally convex topological linear spaces. Proceedings of the National Academy of Sciences of the United States of America, 38(2), 121.
7. Ha, C. W. (1980). Minimax and fixed point theorems. Mathematische Annalen, 248(1), 73-77.
8. Stoer, J. (1963). Duality in nonlinear programming and the minimax theorem. Numerische Mathematik, 5(1), 371-379.