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# Some Application of Cauchy- Riemann Equation to Complex Analysis

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## *1 .Abstract*

The equation $x^2 = -1$ has no solution in the set of real number because the square of every real number is either positive or zeros. Therefore we feel the necessity to extend the system of real numbers. We all know that this defect is remedied by introducing complex numbers.

Baron Augustin - **Louis Cauchy** ; 21 August 1789 – 23 May 1857 ) was a French mathematician reputed as a pioneer of analysis. He was one of the first to state and prove theorems of calculus rigorously , rejecting the heuristic principle of the generality of Algebra of earlier authors. He almost singlehandedly founded complex analysis and the study of permutation groups in abstract Algebra . A profound mathematician , Cauchy had a great influence over his contemporaries and successors. His writings range widely in mathematics and mathematical physics.

"More concepts and theorems have been named for Cauchy than for any other mathematician (in elasticity alone there are sixteen concepts and theorems named for Cauchy)."Cauchy was a prolific writer; he wrote approximately eight hundred research articles and five complete textbooks. He was a devout Roman Catholic, strict Bourbon royalist, and a close associate of the Jesuit order.

Georg Friedrich **Bernhard Riemann** (September 17, 1826 – July 20, 1866) was an influential German mathematician who made lasting contributions to analysis, number theory, and differential geometry, some of them enabling the later development of general relativity.

In his dissertation, he established a geometric foundation for complex analysis through Riemann surfaces, through which multi-valued functions like the logarithm (with infinitely many sheets ) or the square root ( with two sheets ) could become one-to-one functions. Complex functions are harmonic functions (that is, they satisfy Laplace's equation and thus the Cauchy-Riemann equations) on these surfaces and are described by the location of their singularities and the topology of the surfaces.

In this project we are going to study one of important result named after those two great mathematicians “The Cauchy –Riemann Equation “ related to complex Analysis , for that reason we have as title of this work **“Some Application of Cauchy- Riemann Equation to Complex Analysis ”**

to do this work we subdivided it into 4 chapter as :

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Chapter one the Introduction to Complex Number, and the second chapter we have Introduction to complex Analysis. In the third chapter we the Cauchy Riemann Equation.

Finally in the last chapter we have some application of Cauchy Riemann Equation.

## 2. Historical remarks

This system of equations first appeared in the work of (**Jean Le Rond d'Alembert 1752**), he said that : for any two orthogonal directions s and n , with the same mutual orientation as the x and y -axes, in the form:

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n}\ , \quad \frac{\partial u}{\partial n} = -\frac{\partial v}{\partial s}$$

Later, **Leonhard Euler** connected this system to the analytic functions (*Euler* 1797) then **Cauchy** (1814) used these equations to construct his theory of functions. Riemann's dissertation (**Riemann** 1851) on the theory of functions appeared in 1851.

## 3. *Cauchy–Riemann equations*

In the field of complex analysis in mathematics, the ***Cauchy–Riemann equations*** , named after **Augustin Cauchy** and **Bernhard Riemann** , consist of a system of two partial differential equations which, together with certain continuity and differentiability criteria, form a necessary and sufficient condition for a complex function to be complex differentiable , that is holomorphic .

The Cauchy–Riemann equations on a pair of real-valued functions of two real variables *u(x,y)* and *v(x,y)* are the two equations:

$$\text{(a)} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\text{(b)} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Typically u and v are taken to be the real and imaginary parts respectively of a complex-valued function of a single complex variable

$z = x + iy$ , $f(x + iy) = u(x, y) + iv(x, y)$.

Suppose that u and v are real-differentiable at a point in an open subset of C (C is the set of complex numbers), which can be considered as functions from $R^2$ to R. This implies that the partial derivatives of u and v exist (although they need not be continuous) and we can approximate small variations of f linearly.

Then $f = u + iv$ is complex-differentiable at that point *if and only if* the partial derivatives of *u* and *v* satisfy the Cauchy–Riemann equations (a) and (b) at that point.

The sole existence of partial derivatives satisfying the Cauchy–Riemann equations is not enough to ensure complex differentiability at that point.

It is necessary that *u* and *v* be real differentiable, which is a stronger condition than the existence of the partial derivatives, but it is not necessary that these partial derivatives be continuous.[26]

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**Holomorphy** is the property of a complex function of being differentiable at every point of an open and connected subset of $C$ (this is called a domain in $C$).

Consequently, we can assert that a complex function $f$, whose real and imaginary parts $u$ and $v$ are real-differentiable functions, is holomorphic if and only if, equations (a) and (b) are satisfied throughout the domain we are dealing with.The reason why Euler and some other authors relate the Cauchy–Riemann equations with analyticity is that a major theorem in complex analysis says that holomorphic functions are analytic and vice versa.

This means that, in complex analysis, a function that is complex-differentiable in a whole domain (holomorphic) is the same as an analytic function.

## 4. Complex differentiability

Suppose that

$$f(z) = u(z) + i \cdot v(z)$$

is a function of a complex number $z$. Then the complex derivative of $f$ at a point $z_0$ is defined by

$$\lim_{\substack{h\to 0\\ h\in\mathbf{C}}} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$$

provided this limit exists.

If this limit exists, then it may be computed by taking the limit as $h \to 0$ along the real axis or imaginary axis; in either case it should give the same result.

Approaching along the real axis, one finds

$$\lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{f(z_0+h)-f(z_0)}{h} = \frac{\partial f}{\partial x}(z_0).$$

On the other hand, approaching along the imaginary axis,

$$\lim_{\substack{h\to 0\\ h\in\mathbf{R}}} \frac{f(z_0+ih)-f(z_0)}{ih} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

The equality of the derivative of $f$ taken along the two axes is

$$i\frac{\partial f}{\partial x}(z_0) = \frac{\partial f}{\partial y}(z_0),$$

which are the Cauchy–Riemann equations (2) at the point $z_0$.

Conversely, if $f : C \to C$ is a function which is differentiable when regarded as a function on $R^2$, then f is complex differentiable *if and only if* the Cauchy–Riemann equations hold.

In other words, if $u$ and $v$ are real-differentiable functions of two real variables, obviously $u + iv$ is a ( complex-valued ) real - differentiable function.but $u + iv$ is complex-differentiable *if and only if* the Cauchy–Riemann equations hold. Indeed, following Rudin (1966),

suppose f is a complex function defined in an open set $\Omega \subset C$.

Then, writing $z = x + iy$ for every $z \in \Omega$, one can also regard $\Omega$ as an open subset of $R^2$, and f as a

ISSN No: 2348-4845
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A Peer Reviewed Open Access International Journal

function of two real variables x and y, which maps $\Omega \subset R^2$ to $C$.

We consider the Cauchy–Riemann equations at $z = 0$ assuming $f(z) = 0$, just for notational simplicity – the proof is identical in general case. So assume $f$ is differentiable at 0, as a function of two real variables from $\Omega$ to $C$. This is equivalent to the existence of two complex numbers α and β (which are the partial derivatives of $f$ such that we have the linear approximation

$$f(z) = \alpha x + \beta y + \eta(z) z$$

where $z = x + iy$ and $\eta(z) \to 0$ as $z \to z0 = 0$.

Since $z + \bar{z} = 2x$ *and* $z - \bar{z} = 2iy$ , the above can be re-written as

$$f(z) = \frac{\alpha - i\beta}{2} z + \frac{\alpha + i\beta}{2} \bar{z} + \eta(z) z$$

Defining the two Wirtinger derivatives as

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

the above equality can be written as

$$\frac{f(z)}{z} = \left(\frac{\partial f}{\partial z}\right)(0) + \left(\frac{\partial f}{\partial \bar{z}}\right)(0) \cdot \frac{\bar{z}}{z} + \eta(z), \quad (z \neq 0).$$

For real values of z,we have $\frac{\bar{z}}{z} = 1$ and for purely imaginary $z$ we have $\frac{\bar{z}}{z} = -1$ hence $f(z)/z$ has a limit at 0 .

(i.e. $f$ is complex differentiable at 0 if and only if $\left(\frac{\partial f}{\partial z}\right)(0) = 0$ ).

But this is exactly the Cauchy–Riemann equations.

Thus $f$ is differentiable at 0 if and only if the Cauchy –Riemann equations hold at 0.

## 5. *Physical interpretation*

One interpretation of the Cauchy–Riemann equations (Pólya & Szegő 1978) does not involve complex variables directly.

Suppose that u and v satisfy the Cauchy–Riemann equations in an open subset of $R^2$, and consider the vector field

$$\bar{f} = \begin{bmatrix} u \\ -v \end{bmatrix}$$

regarded as a (real) two-component vector. Then the second Cauchy–Riemann equation (b) asserts that $\bar{f}$ is irrotational (its curl is 0):

$$\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

The first Cauchy–Riemann equation (a) asserts that the vector field is solenoidal (or divergence-free):

$$\frac{\partial u}{\partial x} + \frac{\partial(-v)}{\partial y} = 0.$$

ISSN No: 2348-4845
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Owing respectively to Green's theorem and the divergence theorem, such a field is necessarily a conservative one, and it is free from sources or sinks, having net flux equal to zero through any open domain without holes. (These two observations combine as real and imaginary parts in Cauchy's integral theorem.)

- In fluid dynamics, such a vector field is a potential flow (Chanson 2007).
- In magneto statics, such vector fields model static magnetic fields on a region of the plane containing no current.
- In electrostatics, such vector fields model static electric fields in a region of the plane containing no electric charge.

**5. Proof of the Cauchy-Riemann equations in rectangular coordinates.**

Let $f(z)$ be a complex-valued function of a complex variable

$$z = x + iy$$

Let $f(z) = u(z) + iv(z)$

Define the derivative of f(z) to be

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Then we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(z+\Delta z) - u(z)}{\Delta z} + i\frac{v(z+\Delta z) - v(z)}{\Delta z}$$

Now we let $\Delta z = \Delta x + 0\,i$

This gives

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

Recalling the definition of a partial derivative from vector calculus shows that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Now we return to our previous equation in $u, v, and\ z$ and let

$\Delta z = 0 + \Delta y$

This gives

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

Again recalling the definition of partial derivative, we see that

$$f'(z) = -i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

OR

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Observe that these are both equations for f'(z)! Thus we set the real and imaginary parts equal to one another and obtain the famous Cauchy-Riemann equations in rectangular form.

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$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad and \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

**6.Proof of the Cauchy-Riemann equations in Polar Coordinates**

Proof of the Cauchy-Riemann Equations in polar coordinates.

If we let

$$z = r\, e^{i\theta}$$

Then we have the following important relationships which are familiar from analytic geometry

$$x = r\cos\theta$$

$$y = r\sin\theta$$

We proceed to finding polar equivalents of our partial derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial \theta} * \frac{-1}{r sin\theta}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} = \frac{\partial v}{\partial r} * \frac{1}{sin\theta}$$

Since we know these expressions are equal from the rectangular forms

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Continuing with the next set gives

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = \frac{\partial u}{\partial r} * \frac{1}{sin\theta}$$

$$-\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = -\frac{\partial v}{\partial \theta} * \frac{-1}{r sin\theta}$$

Again, we know these equations are equal, so

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

Thus we have the Cauchy-Riemann Equations for polar coordinates as well!

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad and \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

***7 . Applications***

**7.1 Laplace's equation with Complex variables**

Let's look at Laplace's equation in 2D, using Cartesian coordinates:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

It has no real characteristics because its discriminate is negative ($B^2 - 4AC = -4$). But if we ignore this technicality and allow ourselves a complex change of variables, we can benefit from the same structure of solution that worked for the wave equation. Introduce

$$\eta = x + iy\ ;\ x = \frac{\eta + \xi}{2}\ ;\ \xi = x - iy\ ; y = \frac{\eta - \xi}{2i}.$$

Then the chain rule gives

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} \quad ; \quad \frac{\partial}{\partial y} = i\,(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi})$$

and the PDE becomes $\quad 4\frac{\partial}{\partial \eta}\frac{\partial f}{\partial \xi} = 0$

whose solution is straightforward:

$$f = p(\eta) + q(\xi) = p(x + iy) + q(x - iy).$$

Here p and q are differentiable complex functions; and assuming we wanted a real solution to the original (real) PDE.

we have an additional constraint that the sum of the two functions must have no imaginary part.

We can formalize this in more standard notation:

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A Peer Reviewed Open Access International Journal

if we use the $(x, y)$ plane to represent the complex plane in the usual way, we introduce the complex variable $z = x + iy$.

Then its complex conjugate is $\bar{z} = x - iy$

and the solution we have just found is $f = p(z) + q(\bar{z})$.

### 7.2 Composition of Analytic functions

The composition of two analytic functions is analytic (providing, of course, the relevant domains are correctly specified):

If $f: D_1 \to D_2$ and $g: D_2 \to D_3$ are both analytic,

then the composed function $g \circ f : D_1 \to D_3$ is also analytic on $D_1$.

This has important ramifications for the solution of Laplace's equation in odd-shaped domains or with boundary conditions which are unsuitable for separation of variables. Suppose we are trying to find a real function u satisfying

$$\nabla^2 \mathbf{u} = 0 \text{ on } \mathbf{D_1} \text{ with } \mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{y}) \text{ on } \mathbf{D_1}.$$

Of course this is equivalent to finding an analytic function $f(z)$ on $D_1$ whose real part satisfies the boundary condition on $\partial D_1$. If $D_1$ is an awkward shape, and we can find an analytic function w(z) which maps it to a more helpful domain D2, then we can define

$$f = g \circ w \qquad f(z) = g(w(z))$$

and we are now looking for an analytic function g defined on $D_2$ such that

$$\text{Real}\,(g(w(z))) = u(z) \text{ on } \partial D_1. \text{Real}\,(g(w)) = u(z(w)) \text{ on } \partial D_2.$$

### 3. How to show a function is analytic by using Cauchy –Riemann Equation

Showing that a function is analytic within an open region is a lot simpler than it first appears. The definition of analyticity requires that every point within the region the function is differentiable. Using the Cauchy-Riemann equations we only have to find first partial derivatives to get the terms we need to show the function is analytic.

**Example** Let $f(z) = e^{iz}$, show that $f(z)$ is entire (analytic everywhere)

Solution Firstly we need to get the function into the form

$$f(z) = u(x, y) + iv(x, y).$$

We do this using the definition of the exponential and Eulers equation.

$$f(z) = e^{iz^2} = e^{i(x^2-y^2+ixy)}$$

$$= e^{i(x^2-y^2)-xy}$$

$$= e^{-xy} . e^{i(x^2-y^2)}$$

$$f(z) = e^{-xy}[\cos(x^2 - y^2) + i\sin(x^2 - y^2)]$$

$$f(z) = e^{-xy}\cos(x^2 - y^2) + ie^{-xy}\sin(x^2 - y^2).$$

So now we have split the function into real and imaginary part, we get the function into the form

$$u(x, y) = e^{-xy}\cos(x^2 - y^2),$$

$$v(x, y) = e^{-xy}\sin(x^2 - y^2).$$

Now we use partial differentiation to get

$$\frac{\partial u}{\partial x} = -e^{-xy}\sin(x^2 - y^2)$$

$$\frac{\partial v}{\partial y} = -e^{-xy}\sin(x^2 - y^2)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial y} = -e^{-xy}\cos(x^2 - y^2)$$

$$\frac{\partial v}{\partial x} = e^{-xy}\cos(x^2 - y^2)$$

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$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So the function is analytic whenever these equations are satisfied and continuous, which is for all $x$ and for all $y$. So the function is entire.

### *7.3. Determination of the conjugate function*

If $f(z) = u + iv$ is an analytic function , $u$ and $v$ are called conjugaten functions

being given one of these say , $u(x, y)$ , to determine the other $v(x, y)$

we have $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$ ,

since $v$ is a function of $x$ *and* $y$

or $dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$,……………………(1)

by Cauchy _Riemman Equations the equation (1) is of the form

$dv = Mdx + Ndy,$ where $M = -\frac{\partial u}{\partial y}$ , $N = \frac{\partial u}{\partial x}$.

Now $\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

Since $f(z)$ is analytic nfunction therefore u is a harmonic function

i.e. It satisfies Laplace's equation

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$

so that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus equation (1) satisfies the condition of exact differential equation . Therefore $v$ can be determined by integrating (1)

### *7.4 Milne _Thomson's method*

we have $f(z) = u(x, y) + iv(x, y)$ and $z = x + iy$

then $x = \frac{1}{2}(z + \bar{z})$, $y = \frac{1}{2i}(z - \bar{z})$.

we can write

$f(z) = u[\frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})] + iv[\frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})]$ ……..(1)

This realation can be regarded a formal identity in two independent variables z and $\bar{z}$

Putting $z = \bar{z}$ in (1) ,

we get $f'(z) = \frac{\partial f}{\partial x} = u_x + iv_x = u_x - iv_y$

by using Cauchy –Riemaan Equation ,let $u_x = \emptyset_1(x, y), u_y = \emptyset_2(x, y)$

then $f'(z) = \emptyset_1(x, y) - i\emptyset_2(x, y) = \emptyset_1(z, 0) - i\emptyset_2(z, 0)$.

Integrating , we get

$$f(z) = \int \psi_1(z, 0)dz + i\int \psi_2(z, 0)dz + c,$$

Where $v_y = \psi_1(x, y), v_x = \psi_2(x, y)$

### *7.5 Harmonic Conjugate of a function*

**Theorem**

If $f(z) = u + iv$ is analytic in a domain $D$ , then v is the harmonic conjugate of $u$.

conversely , if $v$ is the harmonic conjugate of u in a domain $D$ , then $f(z) = u + iv$ is analytic in $D$.

Proof : since $f(z) = u + iv$ is analytic in $D$ ,

Cauchy –Riemann equations are satisfied , i.e

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiating partially with respect to y and adding , we get

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$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

therefore $u$ *and* $v$ are harmonic functions in $D$ and v is the harmonic conjugate of u because $u$ *and* $v$ satisfy Cauchy-Riemmann Equations.

Conversely ,let $v$ be the harmonic conjugate of $u$.then by the definition of the harmonic conjugate of $u$.

Then by the definition of the harmonic conjugate of $u, v$ is harmonic and Cauchy –Riemman equations $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are all continuous functions.

Hence , $f(z)=u+iv$ is analytic in $D$.

**Remark** :It is very important to note that if $v$ is a harmonic conjugate of $u$ in some domain $D$ , then it is always not true nthaat $u$ is also the harmonic conjugate of $v$ in $D$.

We illustrate this by the following example:

Let $u=x^2-y^2$ and $v=2xy$

Then $f(z)=u+iv$ is analytic in $D$ as shown below

we have $\frac{\partial u}{\partial x}=2x, \frac{\partial u}{\partial y}=-2y,$ $\frac{\partial v}{\partial x}=2y, \frac{\partial v}{\partial y}=-2y$

we see that $\frac{\partial v}{\partial x}\neq\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}\neq-\frac{\partial u}{\partial x}$.

Thus if $\emptyset(z)=v+iu$ then $v$ *and* $u$ do not satisfy Cauchy-Riemann equations.

Therefore $\emptyset(z)$ is not analytic in $D$ .

Hence ,$u$ is not the harmonic conjugate of $v$.

### *7.6 Potential flow*

**Potential flow in two dimensions**

Potential flow in two dimensions is simple to analyze using conformal mapping, by the use of transformations of the complex plane.

However, use of complex numbers is not required, as for example in the classical analysis of fluid flow past a cylinder. It is not possible to solve a potential flow using complex numbers in three dimensions.

The basic idea is to use a holomorphic (also called analytic) or meromorphic function $f$, which maps the physical domain $(x, y)$ to the transformed domain $(\varphi, \psi)$. While $x, y, \varphi$ and $\psi$ are all real valued, it is convenient to define the complex quantities

$$z=x+iy \qquad \text{and} \qquad w=\varphi+i\psi.$$

Now, if we write the mapping $f$ as

$$f(x+iy)=\varphi+i\psi \qquad \text{or} \qquad f(z)=w.$$

Then, because f is a holomorphic or meromorphic function, it has to satisfy the Cauchy-Riemann equations

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y}, \qquad \frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial x}.$$

The velocity components $(u, v)$, in the $(x, y)$ directions respectively, can be obtained directly from $f$ by differentiating with respect to z. That is

$$\frac{df}{dz}=u-iv$$

So the velocity field $v = (u, v)$ is specified by

$$u=\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y}, \qquad v=\frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial x}.$$

Both $\varphi$ *and* $\psi$ then satisfy Laplace's equation :

$$\Delta\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0 \qquad \text{and} \qquad \Delta\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0.$$

So $\varphi$ can be identified as the velocity potential and $\psi$ is called the stream function. Lines of constant $\psi$ are known as streamlines and lines of constant $\varphi$ are known as equipotential lines.

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Streamlines and equi potential lines are orthogonal to each other, since

$$\nabla\emptyset.\nabla\psi = \frac{\partial\emptyset}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\emptyset}{\partial y}\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} = 0$$

Thus the flow occurs along the lines of constant $\psi$ and at right angles to the lines of constant $\varphi$. It is interesting to note that $\Delta\psi = 0$ is also satisfied.

This relation being equivalent to $\nabla \times v = 0$. So the flow is irrotational.

The automatic condition $\frac{\partial^2\Psi}{\partial x\,\partial y} = \frac{\partial^2\Psi}{\partial y\,\partial x}$

then gives the incompressibility constraint $\nabla \cdot v = 0$.[35]

**Examples of Two-dimensional potential flows**

**(i) General considerations**

Any differentiable function may be used for $f$. The examples that follow use a variety of elementary functions; special functions may also be used.

Note that multi-valued functions such as the natural logarithm may be used, but attention must be confined to a single Riemann surface.

**(ii) Power laws**

In case the following power-law conformal map is applied, from $z = x + i$ to $w = \varphi + i\psi$

$$w = Az^n$$

then, writing z in polar coordinates as $z = x + iy = re^{i\theta}$, we have

$$\varphi = Ar^n \cos n\theta \quad \text{and} \quad \psi = Ar^n \sin n\theta$$

In the figure (4.1) to the right examples are given for several values of $n$. The black line is the boundary of the flow, while the darker blue lines are streamlines, and the lighter blue lines are equi-potential lines. Some interesting powers $n$ are:[11]

$n = \frac{1}{2}$ : this corresponds with flow around a semi-infinite plate,

$n = \frac{2}{3}$: flow around a right corner,

$n = 1$ : a trivial case of uniform flow,

$n = 2$ : flow through a corner, or near a stagnation point, and

$n = -1$ : flow due to a source doublet

The constant A is a scaling parameter: its absolute value |A| determines the scale, while its argument $arg(A)$ introduces a rotation (if non-zero).

**Power laws with n = 1: uniform flow**

If $w = Az^1$, that is, a power law with $n = 1$, the streamlines (i.e. lines of constant $\psi$) are a system of straight lines parallel to the $x - axis$. This is easiest to see by writing in terms of real and imaginary components:

$$f(x+iy) = A \times (x+iy) = Ax + i \cdot Ay$$

thus giving $\varphi = Ax$ and $\psi = Ay$. This flow may be interpreted as uniform flow parallel to the x-axis.

**Power laws with n = 2**

If $n = 2$, then $w = Az^2$ and the streamline corresponding to a particular value of $\psi$ are those points satisfying

$$\psi = Ar^2 \sin 2\theta,$$

which is a system of rectangular hyperbolae. This may be seen by again rewriting in terms of real and imaginary components. Noting that $\sin 2\theta = 2\sin\theta\,\cos\theta$ and rewriting

$$\sin\theta = y/r \quad , \quad \cos\theta = x/r$$

it is seen (on simplifying) that the streamlines are given by

$$\psi = 2Axy.$$

The velocity field is given by ∇ φ, or

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial\varphi}{\partial x} \\ \frac{\partial\varphi}{\partial y} \end{pmatrix} = \begin{pmatrix} +\frac{\partial\psi}{\partial y} \\ -\frac{\partial\psi}{\partial x} \end{pmatrix} = \begin{pmatrix} +2Ax \\ -2Ay \end{pmatrix}.$$

In fluid dynamics, the flow field near the origin corresponds to a stagnation point. Note that the fluid at the origin is at rest (this follows on differentiation of $f(z) = z^2$ at $z = 0$).

The $\psi = 0$ streamline is particularly interesting: it has two (or four) branches, following the coordinate axes, i.e. $x = 0$ and $y = 0$. As no fluid flows across the x-axis, it (the x-axis) may be treated as a solid boundary. It is thus possible to ignore the flow in the lower half-plane where $y < 0$ and to focus on the flow in the upper half-plane.

With this interpretation, the flow is that of a vertically directed jet impinging on a horizontal flat plate.

The flow may also be interpreted as flow into a 90 degree corner if the regions specified by (say) $x, y < 0$ are ignored.

**Power laws with n = 3**

If $n = 3$, the resulting flow is a sort of hexagonal version of the $n = 2$ case considered above. Streamlines are given by, $\psi = 3x^2y - y^3$ and the flow in this case may be interpreted as flow into a 60 degree corner.

**Power laws with n = −1**

If $n = -1$, the streamlines are given by

$$\psi = -\frac{A}{r}\sin\theta.$$

This is more easily interpreted in terms of real and imaginary components:

$$\psi = \frac{-Ay}{r^2} = \frac{-Ay}{x^2+y^2},$$

$$x^2 + y^2 + \frac{Ay}{\psi} = 0,$$

$$x^2 + \left(y + \frac{A}{2\psi}\right)^2 = \left(\frac{A}{2\psi}\right)^2.$$

Thus the streamlines are circles that are tangent to the $x-axis$ at the origin.

The circles in the upper half-plane thus flow clockwise, those in the lower half-plane flow anticlockwise.

Note that the velocity components are proportional to $r^{-2}$; and their values at the origin is infinite.

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This flow pattern is usually referred to as a doublet and can be interpreted as the combination of source-sink pair of infinite strength kept at an infinitesimally small distance apart. The velocity field is given by

$$(u,v)=\left(\frac{\partial\psi}{\partial y},-\frac{\partial\psi}{\partial x}\right)=\left(A\frac{y^2-x^2}{(x^2+y^2)^2},-A\frac{2xy}{(x^2+y^2)^2}\right).$$

or in polar coordinates:

$$(u_r,u_\theta)=\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta},-\frac{\partial\psi}{\partial r}\right)=\left(-\frac{A}{r^2}\cos\theta,-\frac{A}{r^2}\sin\theta\right).$$

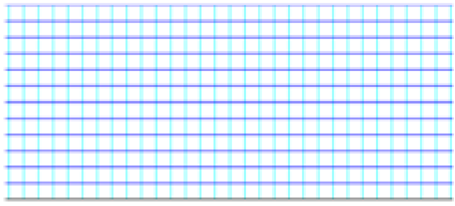
Power laws with $n = -2$: quadrupole

If n = −2, the streamlines are given by

$$\psi=-\frac{A}{r^2}\sin(2\theta).$$

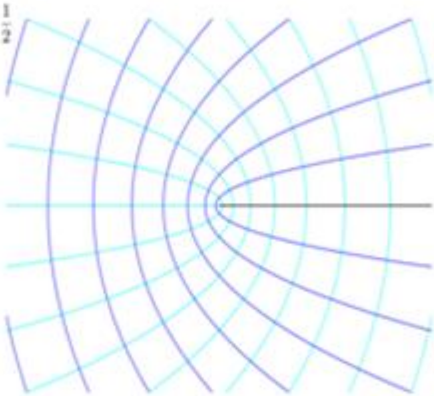
This is the flow field associated with a quadrupole.[36]

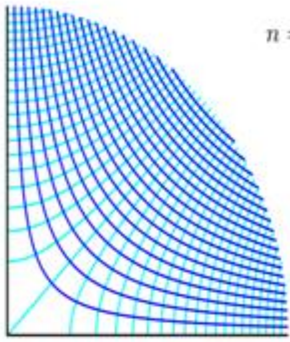
**Examples of conformal maps** for the power law $w = Az^n$, for different values of the power $n$. Shown is the $z-plane$, showing lines of constant potential $\varphi$ and streamfunction $\psi$, while $w = \varphi + i\psi$.



$n = 1$

$n = \frac{1}{2}$





$n = 2$

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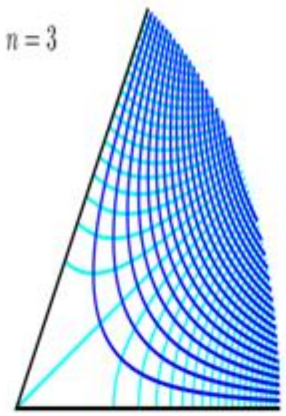


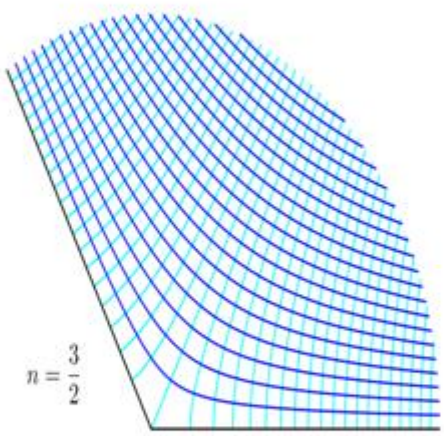


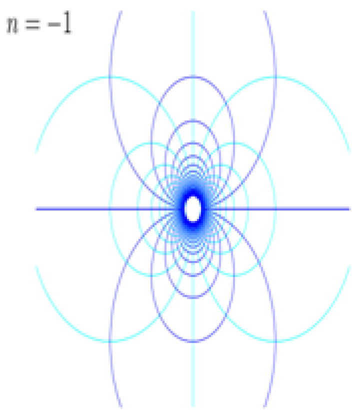


## *7. Conclusion*

Cauchy-Riemann equations in complex analysis, so called in honour of Augustin Cauchy and Bernhard Riemann, are two partial differential equations expressing a necessary and sufficient condition for a function (of a complex variable, with values complex) differentiable at a point in the real sense is differentiable in the complex sense at this point.

In other words, what are the requirements to add to differentiability in the real sense for differentiability in the complex sense.

When the function is differentiable in the real sense in every respect an open, these equations express a necessary and sufficient condition for it to be holomorphic on the open.

Considered a function $f: U \to C$ of a complex variable, defined on an open subset U of the complex plane $\mathbb{C}$. Here we use the following notations:

- The complex variable $z$ is denoted $x + iy$ where $x, y$ are real;
- The real and imaginary parts of $f(z) = f(x + iy)$ are respectively denoted $P(x, y) and\ Q(x, y)$ that is to say: $f(z) =$

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$P(x,y)+iQ(x,y)$, where $P,Q$ are two real functions of two real variables.

It is important to note that the $\mathbb{C}$- differentiability condition for complex variable functions is much more restrictive than the analogous condition for the real variable functions, The difference is this .

- In $\mathbb{R}$ , there are essentially two ways to approach a point right or left. A real variable function is differentiable at a point if and only if the "rate of increase" admits this point a right limit and a limit to the left with the same value (finite);
- In $\mathbb{C}$ , there is an infinite number of ways to approach a point; each must give rise to a limit (finite) in the "rate of increase", these limits being more equal.

Note the continuity of partial derivatives can be shown (this is an important result of the theory of Cauchy) any holomorphic function on an open $\mathbb{C}$ is analytical , it means that near each point is developable power series; Thus, any holomorphic function is infinitely differentiable, let alone she admits continuous partial derivatives on the open.

The binding nature of the condition of homomorphy is particularly striking when applied Cauchy-Riemann conditions to a real-valued function defined on an open $\mathbb{C}$: both partial derivatives with respect to x and y must then be null and the function must be locally constant .

In other words, a real-valued holomorphic function on a connected open $\mathbb{C}$ necessarily reduced to a constant.

For example, the argument of $z$ function (real , not constant) is not holomorphic. It also verifies easily that the Cauchy-Riemann equations are not satisfied , because its partial derivatives are those of arc tan (y / x). This is obviously the same with the module function $z$ (real, not constant) .

In this project we tied to understand each part of this important result (Cauchy-Riemann equation) by first studying the basics notions of complex analysis and by proving the result on some different coordinate and view to finish by finding some application of this equation ( Cauchy-Riemann equation) .

In conclusion we can say that the main point of this equation in complex analysis is: it can easily used to verify the analytic function of some complex

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function, we know that the analytic of complex function is the main point of complex analysis. Therefore the Cauchy-Riemann equation become also an important result of complex Analysis.

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