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Vibrations in Free and Forced Single Degree of Freedom (SDOF) Systems

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Abstract:

In this chapter, the estimation of vibration in static system for both free and forced vibration of single-degree-of-freedom (SDOF) systems of both Undamped and damped due to harmonic force is considered. The knowledge of the mechanical properties of materials used in mechanical systems devices is critical not only in designing structures such as cantilevers and beams but also for ensuring their reliability. A mechanical system is said to be vibrating when its component part are undergoing periodic oscillations about a central statically equilibrium position. Any system can be caused to vibrate by externally applying forces due to its inherent mass and elasticity. The fundamentals of vibration analysis can be understood by studying the simple mass spring damper model. Indeed, even a complex structure such as an automobile body can be modeled as a "summation" of simple mass–spring–damper models. The mass–spring–damper model is an example of a simple harmonic oscillator. In the theory of vibrations, mode shapes in undamped and damped systems have been clearly explained by mode shape diagrams. This method may be helpful in understanding mode shapes and the response of magnitude, acceleration, time, frequency of the homogeneous beam will be found out at different variables of beam using MATLAB R2013. A vibratory system is a dynamic one which for which the variables such as the excitations (input) and responses (output) are time dependent. The response of a vibrating system generally depends on the initial conditions as well as any form of external excitations. Therefore, analyzing a vibrating system will involve setting up a mathematical model, deriving and solving equations pertaining to the model, interpreting the results and assumptions and reanalyze or redesign if need be.

Keywords: ***vibration, damping elements, forced vibration, free vibration, vibrating systems.***

INTRODUCTION

Vibration is the motion of a particle or a body or system of connected bodies displaced from a position of equilibrium [1]. Most Vibrations are undesirable in machines and structures because they produce increased stresses, energy losses, because added wear, increase bearing loads, induce fatigue, create passenger discomfort in vehicles, and absorb energy from the system [2].

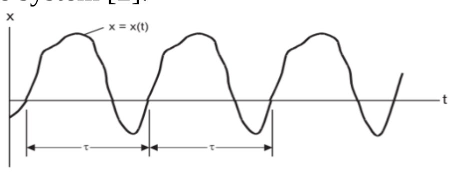

Fig.1 (a): A deterministic (periodic) excitation

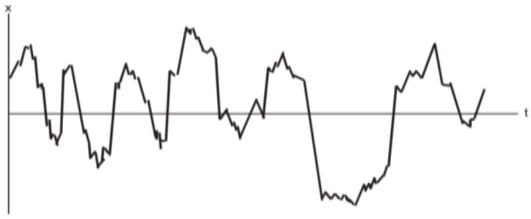

Fig.1 (b) Random excitation

1. CLASSIFICATION OF VIBRATIONS:

Vibrations can be classified into three categories: free, forced, and self-excited. Free vibration of a system is vibration that occurs in the absence of external force. An external force that acts on the system causes forced vibrations [3-4] . In this case, the exciting force

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continuously supplies energy to the system. Forced vibrations may be either deterministic or random (see Fig. 1.1).

1.2 ELEMENTARY PARTS OF VIBRATING SYSTEMS: In general, a vibrating system consists of a spring (a means for storing potential energy), amass or inertia (a means for storing kinetic energy), and a damper (a means by which energy is gradually lost) as shown in Fig. 1.2. An Undamped vibrating system involves the transfer of its potential energy to kinetic energy and kinetic energy to potential energy, alternatively [5]. In a damped vibrating system, some energy is dissipated in each cycle of vibration and should be replaced by an external source if a steady state of vibration is to be maintained.

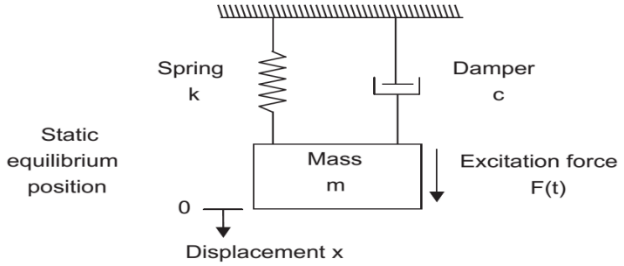

Fig.1.2 Elementary parts of vibrating systems

2. PERIODIC MOTION: When the motion is repeated in equal intervals of time, it is known as *periodic motion*. Simple harmonic motion is the simplest form of periodic motion [6-7]. If $x(t)$ represents the displacement of a mass in a vibratory system, the motion can be expressed by the equation,

$$x = A\cos\omega t = A\cos 2\pi\frac{t}{\tau}$$

Where A is the amplitude of oscillation measured from the equilibrium position of the mass. The repetition time τ is called the *period of the oscillation*, and its reciprocal, $f = 1/\tau$ is called the frequency [8-9]. Any periodic motion satisfies the relationship,

$$x(t)=x(t+\tau)$$

That is,

$$\text{Period } \tau = \frac{2\pi}{\omega}\ \frac{s}{cycle}$$

Frequency,

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}\frac{cycles}{s}, Hz$$

ω is called the circular frequency measured in rad/sec.

3. COMPONENTS OF VIBRATING SYSTEMS

3.1. STIFFNESS ELEMENTS

Sometimes it requires finding out the equivalent spring stiffness values when a continuous system is attached to a discrete system or when there are a number of spring elements in the system [10]. Stiffness of continuous elastic elements such as rods, beams, and shafts, which produce restoring elastic forces, is obtained from deflection considerations [11].

The stiffness coefficient of the rod is given by $\mathbf{k = EA/}\boldsymbol{l}$

The cantilever beam stiffness is $\boldsymbol{k} = \mathbf{3}\boldsymbol{EI/l^3}$

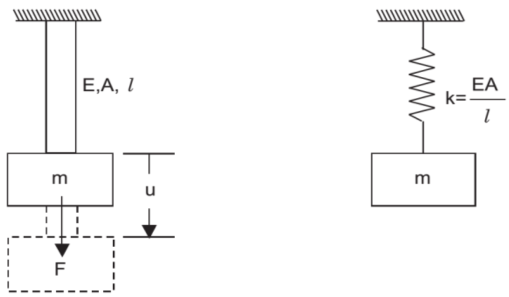

Fig.3.1 (a) longitudinal vibration of rods

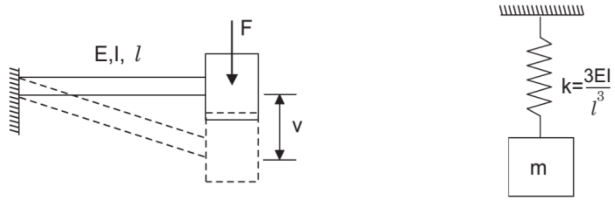

Fig.3.2 (b) Transverse vibration of cantilever beams.

3.1.2. Damping elements:

In real mechanical systems, there is always energy dissipation in one form or another. The process of energy dissipation is referred to in the study of vibration as *damping*. A damper is considered to have neither mass nor elasticity [12-13] . The three main forms of damping are *viscous damping*, *Coulomb* or *dry-friction damping*, and *hysteresis damping*. The

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most common type of energy-dissipating element used in vibrations study is the *viscous damper*, which is also referred to as a *dashpot*. In viscous damping, the damping force is proportional to the velocity of the body. Coulomb or dry-friction damping occurs when sliding contact that exists between surfaces in contact are dry or have insufficient lubrication. In this case, the damping force is constant in magnitude but opposite in direction to that of the motion. In dry-friction damping energy is dissipated as heat [14-15] .

3.2 FREE VIBRATION OF AN UNDAMPED TRANSLATIONAL SYSTEM

The simplest model of a vibrating mechanical system consists of a single mass element which is connected to a rigid support through a linearly elastic mass less spring as shown in Fig. 1.8. The mass is constrained to move only in the vertical direction [16] . The motion of the system is described by a single coordinate *x* (*t*) and hence it has one degree of freedom (DOF).

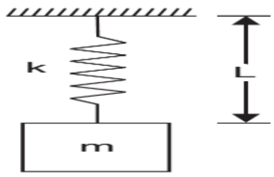


Fig.3.2 Spring mass system.

The equation of motion for the free vibration of an undamped single degree of freedom system can be rewritten as

$$\mathbf{m\ddot{x}(t) + kx(t) = 0}$$

Dividing through by *m*, the equation can be written in the form

$$\mathbf{\ddot{x}(t) + \omega_n^2 x(t) = 0}$$

In which $\omega_n^2 = k/m$ is a real constant. The solution of this equation is obtained from the initial condition

$$\mathbf{x(0) = x_0, \dot{x}(0) = \vartheta_0}$$

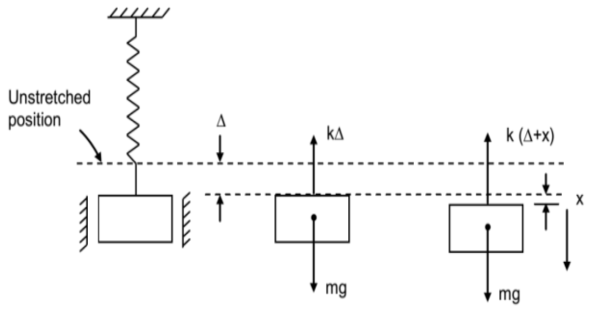
Where x_0 and v_0 are the initial displacement and initial velocity, respectively [17] . The general solution can be written as

$$\mathbf{x(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}}$$

In which *A*1 and *A*2 are constants of integration, both complex quantities. It can be finally simplified as:

$$x(t) = \frac{X}{2}\left[e^{i(\omega_n t-\varphi)} + e^{-i(\omega_n t-\varphi)}\right] = X\cos(\omega_n t-\varphi)$$

4. Free Vibration without damping: The free body diagram of the mass in dynamic condition can be drawn as follows:



(a) Spring Mass (b) Static Condition (c) Dynamic Condition

Fig: 4 (a) Undamped Free Vibration

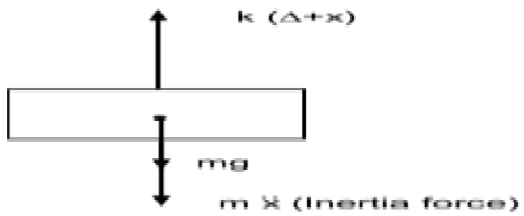


Fig : 4(b) Free Body Diagram

The free body diagram of mass is shown in Figure 7.3 [18-20] . The force equation can be written as follows:

$$m\ddot{x} + mg = k(x + \Delta)$$

Incorporating Eq. (7.1) in Eq. (7.4), the following relation is obtained.

$$\mathbf{m\ddot{x} + k = 0}$$

This equation is same as we got earlier.

Solution of Differential Equation: The differential equation of single degree freedom Undamped system is given by`

$$m\ddot{x} + kx = 0$$

or

$$\ddot{x} + \left(\frac{k}{m}\right)x = 0$$

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When coefficient of acceleration term is unity, the under root of coefficient of x is equal to the natural circular frequency, i.e. 'ω_n'.

$$\omega_n = \sqrt{\frac{k}{m}}$$

Therefore, Eq. becomes,

$$\ddot{x} + \omega_n^2 x = 0$$

The equation is satisfied by functions $\sin w_n t$ and $\cos w_n t$ [19]. Therefore, solution of Eq. can be written as

$$x = A \sin \omega_n t + B \cos \omega_n t$$

Where A and B are constants. These constants can be determined from initial conditions [20] . The system shown in Figure can be disturbed in two ways :

(a) By pulling mass by distance 'X', and

(b) By hitting mass by means of a fast moving object with a velocity \say 'V'.

Considering case (a)

$$t = 0, x = X \text{ and } \dot{x} = 0$$

$$X = B \text{ and } A = 0$$

Therefore,

$$x = X \cos \omega_n t$$

Considering case (b)

$$t = 0, x = 0 \text{ and } \dot{x} = V$$

$$B = 0 \text{ and } A = \frac{V}{\omega_n}$$

Therefore,

$$x = \frac{V}{\omega_n} \sin \omega_n t$$

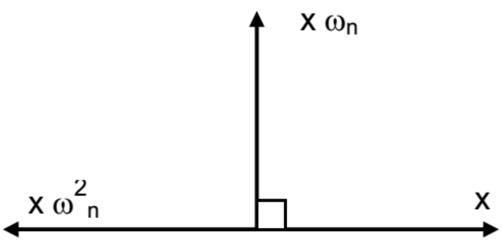

Fig :4(c) plot of displacement and acceleration

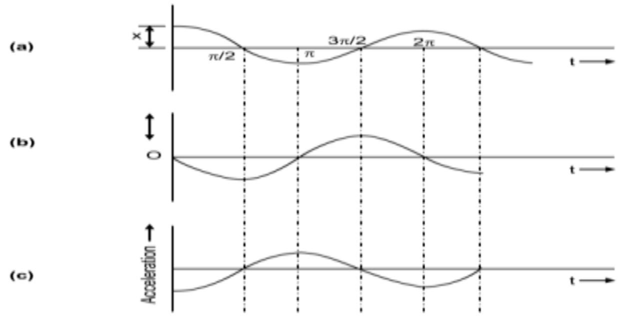

Fig: 4(d) Plots of Displacement, Velocity and acceleration

STEP-1:

4.1. Matlab programming for free vibration without damping:

```
clc
close all
% give mass of the system
m=2;
%give stiffness of the system
k=8;
wn=sqrt(k/m);
%give damping coefficient
c1=1;
u(1)=.3;
udot(1)=.5;
uddot(1)=(-c1*udot(1)-k*u(1))/m;
cc=2*sqrt(k*m);
rho=c1/cc;
wd=wn*sqrt(1-rho^2);
wba=rho*wn;
rhoba=rho/sqrt(1-rho^2);
b0=2.0*rho*wn;
b1=wd^2-wba^2;
b2=2.0*wba*wd;
dt=0.02;
t(1)=0;
for i=2:1500
t(i)=(i-1)*dt;
s=exp(-rho*wn*t(i))*sin(wd*t(i));
c=exp(-rho*wn*t(i))*cos(wd*t(i));
sdot=-wba*s+wd*c;
cdot=-wba*c-wd*s;
```

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```
sddot=-b1*s-b2*c;
cddot=-b1*c+b2*s;
a1=c+rhoba*s;
a2=s/wd;
a3=cdot+rhoba*sdot;
a4=sdot/wd;
a5=cddot+rhoba*sddot;
a6=sddot/wd;
u(i)=a1*u(1)+a2*udot(1);
udot(i)=a3*u(1)+a4*udot(1);
uddot(i)=a5*u(1)+a6*udot(1);
end
figure(1);
plot(t,u,'k');
xlabel(' time');
ylabel(' displacement ');
title(' displacement - time');
figure(2);
xlabel(' time');
ylabel(' velocity');
title(' velocity - time');
figure(3);
plot(t,uddot,'k');
xlabel(' time');
ylabel(' acceleration');
title(' acceleration- time')
```

STEP-2:

4.1.1. Using matlab coding for the above equation the result and plots:

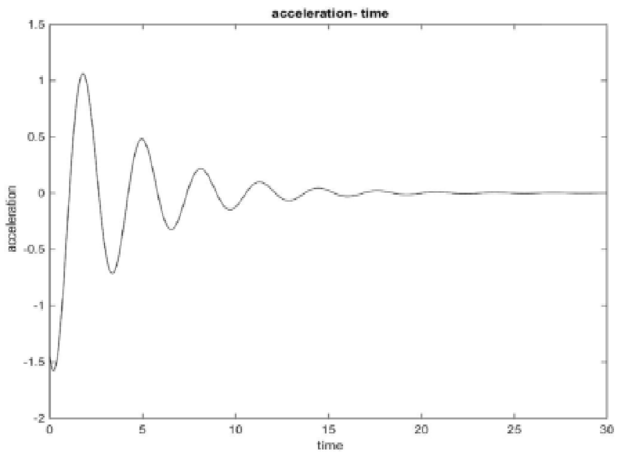

Fig.4.1.1 (a): acceleration-time plot

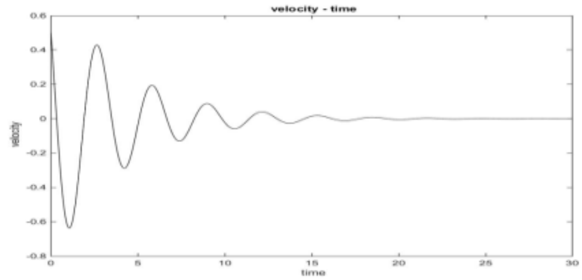

Fig.4.1.2 (b):velocity and time plot

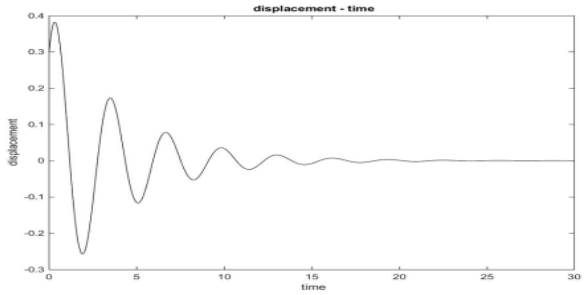

Fig.4.1.3(c):displacement-time plot

5. Free Vibration with damping: In Undamped free vibrations, two elements (spring and mass) were used but in damped third element which is damper in addition to these are used. The three element model is shown in Figure 7.7. [21-22] In static equilibrium

$$k\Delta = mg$$

$$m\ddot{x} = mg - K(x + \Delta) - c\dot{x}$$

Therefore,

$$m\ddot{x} = -Kx - c\dot{x}$$

Or,

$$m\ddot{x} + c\dot{x} + Kx = 0$$

Let,

$$x = Xe^{st}$$

Substituting for x in eq and simplifying it

$$ms^2 + cs + k = 0$$

Or,

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

Therefore,

$$s_{1,2} = -\left(\frac{c}{2m}\right) \pm \frac{1}{2}\sqrt{\left(\frac{c}{m}\right) - 4\left(\frac{k}{m}\right)}$$

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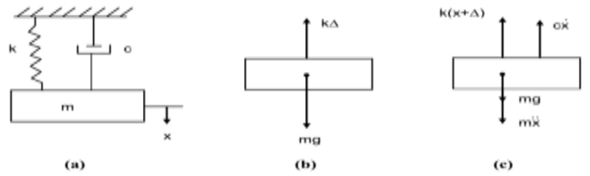

Figure: Damped Free Vibration

The solution of Eq. is given by

$$x = X_1 e^{\left[-\left(\frac{c}{2m}\right)+\frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2-4\left(\frac{k}{m}\right)}\right]t} + X_2 e^{\left[-\left(\frac{c}{2m}\right)-\frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2-4\left(\frac{k}{m}\right)}\right]t}$$

$$= e^{-\left(\frac{c}{2m}\right)t}\left[X_1 e^{\frac{1}{2}\left\{\sqrt{\left(\frac{c}{m}\right)^2-4\left(\frac{k}{m}\right)}\right\}t} + X_2 e^{-\frac{1}{2}\left\{\sqrt{\left(\frac{c}{m}\right)^2-4\left(\frac{k}{m}\right)}\right\}t}\right]$$

The nature of this solution depends on the term in the square root. There are three possible cases [23].

(a) $\left(\frac{c}{m}\right)^2 > 4\left(\frac{k}{m}\right)$ – Overdamped case

(b) $\left(\frac{c}{m}\right)^2 = 4\left(\frac{k}{m}\right)$ – Critically damped case

(c) $\left(\frac{c}{m}\right)^2 < 4\left(\frac{k}{m}\right)$ – Underdamped case

The general solution of the forced vibration without damping is

$$x = x_c + x_p$$

$$x = \underbrace{C_1 \sin \omega_n t + C_2 \cos \omega_n t}_{\text{free vibration (transient)}} + \underbrace{\delta st \sin \omega t/(1-\beta^2)}_{\text{forced vibration (steady state)}}$$

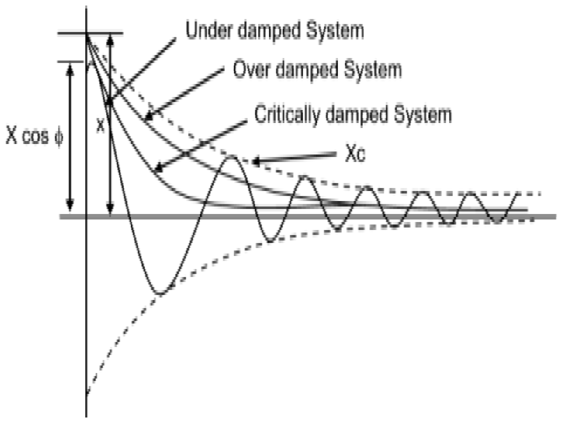

Figure: 5.damped systems

STEP-1:

5.1. Matlab programming for free vibration with damping plot:

```
Clear
m=2;
l0=0.3;
x0=0.1;
v0=0;
k = [288,288,288];
b = [288,2*sqrt(k(2)*m),10];
OM = sqrt(k/m);
NN = b/(2*m);
delta = NN./OM;
t = 0:0.01:1.4;
for i=1:3 OM(i)=sqrt(k(i)/m);
if   i==2,   x=(x0+(v0+x0*OM(i))*t).*exp(-OM(i)*t);
else N(i)=b(i)/(2*m);
za(i)=N(i)^2-OM(i)^2;
lam1(i)=-N(i)+sqrt(za(i));
lam2(i)=-N(i)-sqrt(za(i));
zl1(i)=(v0-lam2(i)*x0)/(lam1(i)-lam2(i));
zl2(i)=(v0-lam1(i)*x0)/(lam1(i)-lam2(i));
x=(zl1(i)*exp(lam1(i)*t))-(zl2(i)*exp(lam2(i)*t));
end
mx(i,:)=x;
end
figure(1)
label1=['overdamped, delta = ' num2str(delta(1))];
label2=['critically damped, delta = ' num2str(delta(2))];
for i=1:3
plot(t,mx(i,:),'k','linewidth',2);
ylabel('displacement x [m]');
else, text(0.6,0.07,label3);xlabel('time t [s]');
end
end
```

STEP-2:

5.2 Using Matlab coding for the above equation the result and plots:

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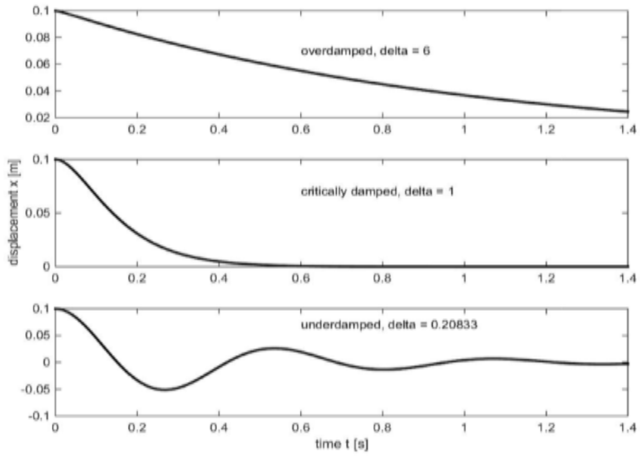

Fig: 5.2.over damped; critically damped; under damped

6. Forced vibration without damping: In many important vibration problems encountered in engineering work, the exciting force is applied periodically during the motion. These are called forced vibrations [24-25]. The most common periodic force is a harmonic force of time such as

$$P = P0 \sin \omega t$$

Where $P0$ is a constant, ω is the forcing frequency and t is the time. The motion is analyzed using Fig.

$$m\ddot{x} + kx = P_0 \sin \omega t$$

The general solution of Eq. 4.2 (non-homogeneous second order differential equation) consists of two parts $x = x_c + x_p$ where x_c = complementary solution,

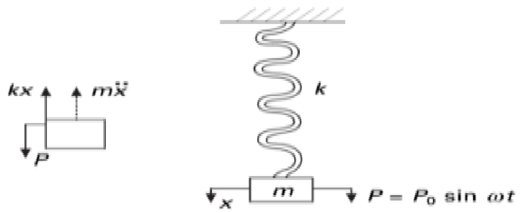

Fig:6. Spring–mass system subjected to harmonic force.

And x_p = particular solution. The complementary solution is obtained by setting right hand side as zero.

STEP-1:

6.1. Matlab programming for forced vibration without dampig plot:

```
clear %assign the initial conditions, mass, damping and stiffness
 x0=0.01;v0=0.0;m=100;c=25;k=1000;
 %compute omega and zeta, display zeta to check if underdamped
 wn=sqrt(k/m);
 z=c/(2*sqrt(k*m));
 z=0.0395;
 % the damped natural frequency
wd=wn*sqrt(1-z.^2);
t=(0:0.01:15*(2*pi/wn));%set the values of time from 0 in
% increments of 0.01 up to 15 periods
x=exp(z*wn*t).*(x0*cos(wd*t)+((v0+z*wn*x0)/wd)*sin(wd*t));
% computes x(t) plot(t,x)%generates a plot of x(t) vs
```

STEP-2:

6.1.2. Using matlab coding for the above equation the result and plots:

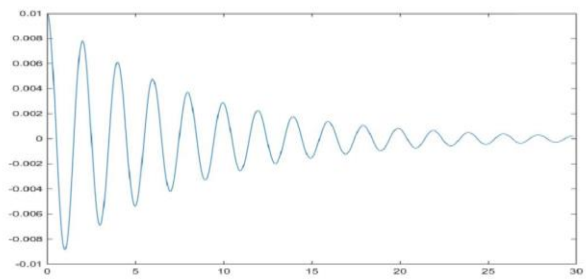

Fig: 6.1.2displacement time plot

7. Forced vibration with damping: Consider a forced vibration of the under-damped system shown in Fig. The dynamic equilibrium equation is written as,

$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega t$$

Equation is a second order non-homogeneous equation and it has both a complementary solution x_c and a particular solution x_p . x_c is same as that for free vibration of an under-damped system [26] .

$$x_c = Xe^{-\rho\omega_n t}(\sin \omega_n \sqrt{1-\rho^2}\, t + \phi)$$

$$\tan \phi = \frac{\omega_d \, x_0}{(v_0 + \rho\,\omega_n \, x_0)}$$

Assume

$$x_p = D \cos \omega t + E \sin \omega t.$$

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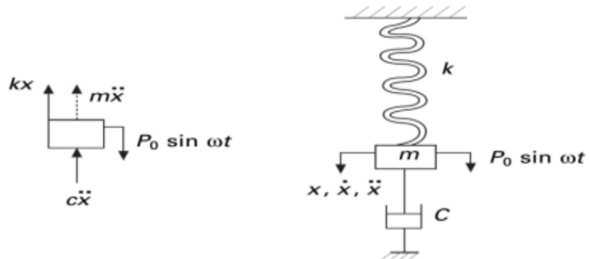

Fig:7. Forced vibration of under-damped system.

Now $x = x_c + x_p$

$$x = Ae^{-\rho\omega_n t}(\sin \omega_n \sqrt{1-\rho^2}\,t + \phi)$$

$$+\frac{P_0}{k}\frac{1}{\sqrt{(1-\beta^2)^2+(2\rho\beta)^2}}\sin(\omega t - \psi)$$

STEP-1:

7.1. Matlab programming for forced vibration with damping plot:

```
 echo off
% sdofxfer.m plotting frequency responses of sdof
model for different
% damping values. Calculates and plots magnitude
and phase for a single
 clf;
 clear all;

 % assign values for mass, percentage of critical
damping, and stiffnesses
% zeta is a vector of damping values from 10% to
100% in steps of 10%
 m = 1;
 zeta = 0.1:0.1:1
 k = 1;
 wn = sqrt(k/m);
 w = logspace(-1,1,400);

% pre-calculate the radians to degree conversion
rad2deg = 180/pi;
% define s as the imaginary operator times the radian
frequency vector
s = j*w;
% define a for loop to cycle through all the damping
values for calculating magnitude and phase
for cnt = 1:length(zeta)
% define the frequency response to be evaluated
xfer(cnt,:) = (1/m) ./ (s.^2 + 2*zeta(cnt)*wn*s +
wn^2);
% calculate the magnitude and phase of each
frequency response
mag(cnt,:) = abs(xfer(cnt,:));
phs(cnt,:) = angle(xfer(cnt,:))*rad2deg;
 end

% define a for loop to cycle through all the damping
values for plotting magnitude
for cnt = 1:length(zeta)
 loglog(w,mag(cnt,:),'k-')
 xlabel('frequency, rad/sec')
 ylabel('magnitude')
 grid
 hold on
 end

 hold off
 grid on
 disp('execution paused to display figure, "enter" to
continue'); pause
 % define a for loop to cycle through all the damping
values for plotting phase
 for cnt = 1:length(zeta)

 semilogx(w,phs(cnt,:),'k-')
title('SDOF frequency response phases for zeta = 0.1
to 1.0 in steps of 0.1')
xlabel('frequency, rad/sec')
 ylabel('magnitude')

 grid
 hold on
 end
 hold off
 grid on
disp('execution paused to display figure, "enter" to
continue'); pause
```

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STEP-2:
Using matlab coding for the above equation the result and plots:

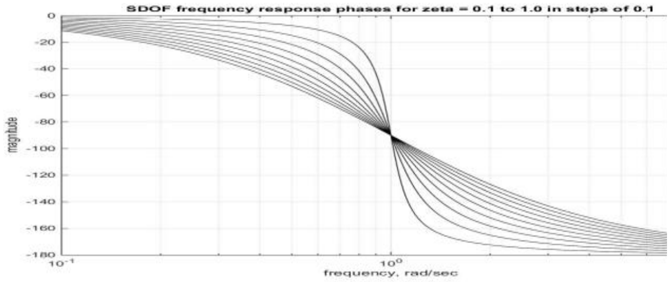

Fig: 7.1(a) SDOF response for magnitude and frequency

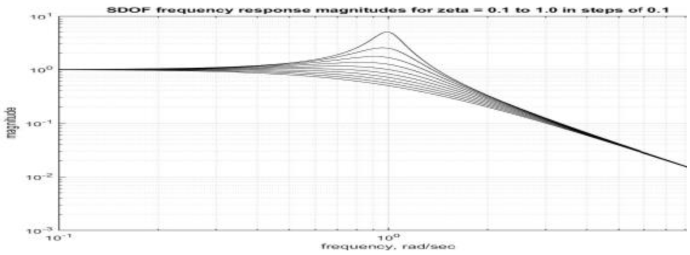

Fig: 7.1(b) SDOF response for magnitude and Frequency in step of 0.1

CONCLUSION:

- The theory of free vibration motion with and without damping was studied and appropriate mathematical model was used to calculate the value of the spring stiffness, K the natural frequency,
- When the large mass was attached, also the value of the natural frequency, with no mass attached. The damping coefficient was also calculated for different damping setting. Damping is very useful and it should be incorporated in the design of systems or mechanism subjected to vibrations and shock as it helps to minimize fatigue and failure. The right damper will reduce stress and deflection.
- The slight structural consideration will show that the amplitude of beam at resonance will be maximum and the problem of failure will arise.
- So, in design considerations the beams taken should be such that there is no resonance for the stability of a structure.
- In this paper, we are using differentiation Method to formulate the equations of motion of homogeneous beams.
- The response of magnitude, acceleration, time, frequency of the homogeneous beam will be found out at different variables of beam using MATLAB R2013.
- The results will be compared with the results found by differentiation method. Using these results, frequency and mode shapes of beam variables will be correlated.

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