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Reducing Initial Value Problem and Boundary Value Problem to Volterra and Fredholm Integral Equation and Solution of Initial Value Problem

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I. Introduction

Origins of Integral Equations

Integral and integro-differential equations arise in many scientific and engineering applications. Volterra integral equations and Volterra integro-differential equations can be obtained from converting initial value problems with prescribed initial values. However, Fredholm integral equations and Fredholm integro-differential equations can be derived from boundary value problems with given boundary conditions. It is important to point out those converting initial value problems to Volterra integral equations, and converting Volterra integral equations to initial value problems are commonly used in the literature. This will be explained in detail in the coming section. However, converting boundary value problems to Fredholm integral equations, and converting Fredholm integral equations to equivalent boundary value problems are rarely used. The conversion techniques will be examined and illustrated examples will be presented. In what follows we will examine the steps that we will use to obtain these integral and integro-differential equations.

Preliminaries

An integral equation is an equation in which the unknown function u(x) appears under an integral sign. A standard integral equation in u(x) is of the form:

$$\mathbf{u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt} \qquad \textbf{(I.1.1)}$$

Where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t called the kernel or the nucleus of the integral equation. The function $u(x)$ that will be determined appears under the integral sign, and it appears inside the integral sign and outside the integral sign as well. The functions $f(x)$ and $K(x,t)$ are given in advance. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed. An integro-differential equation is an equation in which the unknown function $u(x)$ appears under an integral sign and contains an ordinary derivative $u^n(x)$ as well. A standard integro-differential equation is of the for

$$\mathbf{u^n(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt} \qquad \textbf{(I.1.2)}$$

Where$g(x)$, $h(x)$, $f(x)$, λ and the kernel $K(x,t)$ are as prescribed before. Integral equations and integro-differential equations will be classified into distinct

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types according to the limits of integration and the kernel$K(x,t)$. All types of integral equations and integro-differential equations will be classified and investigated in the forthcoming chapters. In this chapter, we will review the most important concepts needed to study integral equations. The traditional methods, such as Taylor series method and the Laplace transform method, will be used in this text. Moreover, the recently developed methods that will be used thoroughly in this text will determine the solution in a power series that will converge to an exact solution if such a solution exists. However, if exact solution does not exist, we use as many terms of the obtained series for numerical purposes to approximate the solution .The more terms we determine the higher numerical

Accuracy we can achieve. Furthermore, we will review the basic concepts for solving ordinary differential equations. Other mathematical concepts, such as Leibnitz rule will be presented.

1.1 Taylor Series

Let f(x) be a function with derivatives of all orders in an interval $[x_0, x_1]$ that contains an interior point a. The Taylor series of f(x) generated at $x = a$is:

$$\mathbf{f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \quad (I.1.3)}$$

Or equivalently

$$\mathbf{f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + .. \quad (I.1.4)}$$

The Taylor series generated by f(x) at a = 0 is called the Maclaurin series and given by:

$$\mathbf{f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n (I.1.5)}$$

That is equivalent to

$$\mathbf{f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + .. + \frac{f^n(0)}{n!}x^n + \quad (I.1.6)}$$

In what follows, we will discuss a few examples for the determination of the Taylor series at $x = 0$.

1.2 Ordinary Differential Equations

In this section we will review some of the linear ordinary differential equations that we will use for solving integral equations. For proofs, existence and uniqueness of solutions, and other details, the reader is advised to use ordinary differential equations texts.

1.2.1 First Order Linear Differential Equations

The standard form of first order linear ordinary differential equation is

$$\mathbf{u' + p(x)u = q(x) \quad (I.1.7)}$$

Where p(x) and q(x) are given continuous functions on $x_0 < x < x_1$. We first determine an integrating factor μ(x) by using the formula:

$$\mathbf{\mu(x) = e^{\left(\int_{x_0}^{x} p(t)\right)dt} \qquad (I.1.8)}$$

Recall that an integrating factor μ(x) is a function of x that is used to facilitate the solving of a differential equation. The solution of **(I.1.7)** is obtained by using the formula:

$$\mathbf{u(x) = \frac{1}{\mu(x)}\left[\int \mu(t)q(t)dt + c\right] (I.1.9)}$$

Where c is an arbitrary constant that can be determined by using a given initial condition

1.2.2 The Series Solution Method

For differential equations of any order, with constant coefficients or with variable coefficient, with x = 0 is an ordinary point, we can use the series solution method to determine the series solution of the differential equation. The obtained series solution may converge the exact solution if such a closed form solution exists. If an exact solution is not obtainable, we may use a truncated number of terms of the obtained series for numerical purposes. Although the series solution can be used for equations with constant coefficients or with variable coefficients, where x = 0 is an ordinary point, but this method is commonly used

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for ordinary differential equations with variable coefficients where x = 0 is an ordinary point. The series solution method assumes that the solution near an ordinary point x = 0 is given by

$$\mathbf{u(x) = \sum_{n=0}^{\infty} a_n x^n,} \qquad \textbf{(I.1.10)}$$

Or by using few terms of the series

$$\mathbf{u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6} \quad \textbf{(I.1.11)}$$

Differentiating term by term gives

$$\mathbf{u'(x) = a_1 + 2a_2x + 3a_3x^2 + ..}$$

$$\mathbf{u''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \cdots}$$

$$\mathbf{u'''(x) = 6a_3 + 24a_4\,x + 60a_5x^2 + \cdots} \qquad \textbf{(I.1.12)}$$

And so on. Substituting u(x) and its derivatives in the given differential equation, and equating coefficients of like powers of x gives a recurrence relation that can be solved to determine the coefficients $a_n, n \geq 0$. Substituting the obtained values of $a_n, n \geq 0$ in the series assumption (I.1.10) gives the series solution. As stated before, the series may converge to the exact solution. Otherwise, the obtained series can be truncated to any finite number of terms to be used for numerical calculations.

The more terms we use will enhance the level of accuracy of the numerical approximation. It is interesting to point out that the series solution method can be used for homogeneous and inhomogeneous equations as well when x = 0 is an ordinary point. However, if x = 0 is a regular singular point of an ODE, then solution can be obtained by Frobenius method that will not be reviewed in this text. Moreover, the Taylor series of any elementary function involved in the differential equation should be used for equating the coefficients. The series solution method will be illustrated by examining the following ordinary differential equations where x = 0 is an ordinary point. Some examples will give exact solutions, whereas others will provide series solutions that can be used for numerical purposes.

1.3 Leibnitz Rule for Differentiation of Integrals

One of the methods that will be used to solve integral equations is the conversion of the integral equation to an equivalent differential equation. The conversion is achieved by using the well-known Leibnitz rule for differentiation of integrals. Let $f(x,t)$ be continuous and $\frac{\partial f}{\partial t}$ be continuous in a domain of the x-t plane that includes the rectangle

$$a \leq x \leq b, t_0 \leq t \leq t_1, \text{and let}$$

$$\mathbf{F(x) = \int_{g(x)}^{h(x)} f(x,t)dt} \quad \textbf{(I.1.13)}$$

then differentiation of the integral in **(1.13)**
Exists and is given by

$$\mathbf{F'(x) = \frac{dF}{dx} = f(x,h(x))\frac{dh(x)}{dx} - f(x,g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x} dt} \quad \textbf{(I.1.14)}$$

If $g(x) = a$ and $h(x) = b$ where a and b are constants, then the Leibnitz rule **(I.1.14)** reduces to

$$\mathbf{F'(x) = \frac{dF}{dx} = \int_a^b \frac{\partial f(x,t)}{\partial x} dt} \quad \textbf{(I.1.15)}$$

Which means that differentiation and integration can be interchanged such as

$$\mathbf{\frac{d}{dx}\int_a^b e^{xt} dt = \int_a^b t\, e^{xt}\, dt} \quad \textbf{(I.1.16)}$$

is interested to notice that Leibnitz rule is not applicable for the Abel's singular integral equation:

$$\mathbf{F(x) = \int_0^x \frac{u(t)}{(x-t)^{\alpha}} d\,t} \qquad \mathbf{0<\alpha<1} \; \textbf{(I.1.17)}$$

The integrand in this equation does not satisfy the conditions that $f(x,t)$ be continuous and $\frac{\partial f}{\partial t}$ be continuous, because it is unbounded at x = t. We illustrate the Leibnitz rule by the following example.

Example 1

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Find $F'(x)$ for the following:

$$F(x) = \int_{sinx}^{cosx} \sqrt{1+t^3}\, dt$$

We can set $g(x) = sinx$ and $h(x) = cosx$. It is also clear that $f(x,t)$ is a

Function of t only. Using Leibnitz rule **(I.1.14)** we find that

$$F'(x) = -sinx\sqrt{1+\cos^3 x} - cosx\sqrt{1+\sin^3 x}$$

1.4 Laplace Transform

In this section we will review only the basic concepts of the Laplace transform method. The details can be found in any text of ordinary differential equations. The Laplace transform method is a powerful tool used for solving differential and integral equations .The Laplace transform changes differential equations and integral equations to polynomial equations that can be easily solved, and hence by using the inverse Laplace transform gives the solution of the examined equation. The Laplace transform of a function f(x), defined for $x \geq 0$, is defined by

$$F(s) = L\{ f(x) \} = \int_0^{\infty} e^{-sx} f(x)\, dx \quad (I.1.18)$$

where s is real, and L is called the Laplace transform operator. The Laplace transform $F(s)$ may fail to exist. If $f(x)$ has infinite discontinuities or if it grows up rapidly, then $F(s)$ does not exist. Moreover, animportant necessary condition for the existence of the Laplace transform $F(s)$ is that $F(s)$ must vanish as s approaches infinity. This means that $\lim_{s\to\infty} F(s) = 0.$ **(I.1.19)**

In other words, the conditions for the existence of a Laplace transform $F(s)$ of any function $f(x)$ are:

1. $F(x)$ is piecewise continuous on the interval of integration $0 \leq x < A$ for any positive A,

2. $f(x)$ is of exponential order e^{ax} as $x \to \infty$, i.e. $|f(x)| \leq Ke^{ax}, x \geq M$

where a is real constant, and K and M are positive constants. Accordingly, the Laplace transform $F(s)$ exists and must satisfy $\lim\limits_{s\to\infty} F(s) = 0.$ **(I.1.20)**

2.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral equation in the form:

$$f(x) = \int_a^b K(x,t)\, u(t)\, dt \quad (I.2.8)$$

This is called Fredholm integral equation of the first kind. However, for Fredholm integral equations of the second kind, the unknown function u(x) appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt \quad (I.2.9)$$

Examples of the two kinds are given by

$$\frac{sinx - xcosx}{x^2} = \int_0^1 \sin(xt)\, u(t)dt \quad (I.2.10)$$

And

$$u(x) = x + \frac{1}{2}\int_{-1}^{1} (x-t)u(t)dt \quad (I.2.11)$$

Respectively.

2.2 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the first kind Volterra integral equations, the unknown function $u(x)$ appears only inside integral sign in the form:

$$f(x) = \int_0^x K(x,t)\, u(t)\, dt. \quad (I.2.12)$$

However, Volterra integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt \quad (I.2.13)$$

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Examples of the Volterra integral equations of the first kind are:

$$xe^{-x} = \int_0^x e^{t-x}\, u(t)\, dt \quad (2.14)$$

and

$$5x^2 + x^3 = \int_0^x (5 + 3x - 3t) u(t) dt \quad (I.2.15)$$

However, examples of the Volterra integral equations of the second kind are

$$u(x) = 1 - \int_0^x u(t)\, dt \qquad (I.2.16)$$

And

$$u(x) = x + \int_0^x (x - t)\, u(t)\, dt \quad (I.2.17)$$

2.3 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modeling of the spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)\, u(t) dt + \lambda_2 \int_a^b K_2(x,t)\, u(t) dt \quad (I.2.18)$$

And

$$u(x,t) = f(x) + \lambda \int_0^t \int_\Omega F\big(x, t, \varepsilon, \tau, u(\varepsilon, t)\big) d\varepsilon\, d\tau\,, (x,t) \in \Omega \times [0\,, T] \quad (I.2.19)$$

Where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on D $=\Omega\times[0,T]$, and Ω is a closed subset of R^n, $n = 1, 2, 3$. It is interesting to note that (I.2.18) contains disjoint Volterra and Fredholm integral equations, whereas (I.2.19) contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind, but will not be examined in this text.

Examples of the two types are given by

$$u(x) = 6x + 3x^2 + 2 - \int_0^x x\; u(t)\, dt - \int_0^1 t\, u\,(t)\; dt, \quad (I.2.20)$$

$$u(x,t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \varepsilon)\, d\varepsilon\, d\tau \quad (I.2.21)$$

2.4 Singular Integral Equations

Volterra integral equations of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t) u(t)\; dt \qquad (I.2.22)$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)\; u(t)\, dt \quad (I.2.23)$$

are called singular if one of the limits of integration $g(x)$, $h(x)$ or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration. In this text we will focus our concern on equations of the form:

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)\, dt \qquad 0 < \alpha < 1 \qquad (I.2.24)$$

or of the second kind:

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t) dt\,, \qquad 0 < \alpha < 1 \qquad (I.2.25)$$

The last two standard forms are called generalized Abel's integral equation and weakly singular integral equations respectively. For $\alpha = \frac{1}{2}$, the equation:

$$f(x)= \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \quad (I.2.26)$$

is called the Abel's singular integral equation. It is to be noted that the kernel in each equation becomes infinity at the upper limit t = x. Examples of Abel's integral equation, generalized Abel's integral equation, and the weakly singular integral equation are given by

$$\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \quad (I.2.27)$$

$$x^3 = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt \quad (I.2.28)$$

And

$$u(x)= 1+\sqrt{x} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt, \quad (I.2.29)$$

Respectively.

2.5 Integro-equation:

2.5.1 Fredholm Integro-Differential Equations

Fredholm integro-differential equations appear when we convert differential equations to integral equations. The Fredholm integro-differential equation contains the unknown function u(x) and one of its derivatives $u^n(x), n \geq 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed as in the Fredholm integral equations. The equation is labeled as integro-differential because it contains differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions. The Fredholm integro-differential equation appears in the form:

$$u^n(x)= f(x) + \lambda \int_a^b K(x,t)\ u(t)\ dt \quad (I.2.30)$$

Where u^n indicates the nth derivative of u(x). other derivatives of less order may appear with u^n at the left side. Examples of the Fredholm integrodifferential equations are given by

$$u'(x)= 1-\frac{1}{3}x+ \int_0^1 xu(t)\ dt, u(0) = 0 \quad (I.2.31)$$

And

$$u''(x)+u'(x) = x - \sin x - \int_0^{\left(\frac{\pi}{2}\right)} xtu(t)dt, \ u(0)=0\ ,\ u(0)=1 \quad (I.2.32)$$

2.5.2 Volterra Integro-Differential Equations

Volterra integro-differential equations appear when we convert initial value problems to integral equations .The Volterra integro-differential equation contains the unknown function u(x) and one of its derivatives $u^n(x), n \geq 1$ inside and outside the integral sign. At least one of the limits of integration in this case is a variable as in the Volterra integral equations. The equation is called integro-differential becausedifferential and integral operators are involved in the same equation. It is important to note that initial conditions should be given for Volterra integro-differential equations to determine the particular solutions. The Volterra integro-differential equation appears in the form:

$$u^n(x) = f(x) + \lambda \int_0^x K(x,t)\ u(t)\ , dt \quad (I.2.33)$$

Whereu^n indicates the nth derivative of u(x). Other derivatives of less order may appear with u^n at the left side. Examples of the Volterra integro-differential equations are given by

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$\mathbf{u'(x)= -1 + \frac{1}{2}x^2 - xe^x - \int_0^x tu(t)dt, \quad u(0)=0,}$ **(I.2.34)**

And

$\mathbf{u''(x)+u'(x) = 1-x(\sin x + \cos x) - \int_0^x tu(t)dt, u(0) = -1, u'(0)=1}$ **(I.2.35)**

2.5.3 Volterra-Fredholm Integro-Differential Equations

The Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators. The Volterra-Fredholm integro-differential equations appear in the literature in two forms, namely

$\mathbf{u^n(x) = f(x) + \lambda_1 \int_0^x K_1(x,t)\ u(t)dt + \lambda_2 \int_a^b K_2(x,t)\, u(t)dt}$ **(I.2.36)**

And

$\mathbf{u^n(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F\big(x,t,\varepsilon,\tau,u(\varepsilon,\tau)\big)d\varepsilon d\,\tau, (x,t) \in \Omega \times [0,T]}$**(I.2.37)**

Where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of R^n,n=1,2,3… It is interesting to note that **(I.2.36)** contains disjoint Volterra and Fredholm integral equations, whereas **(I.2.37)** contains mixed integrals. Other derivatives of less order may appear as well. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind. Initial conditions should be given to determine the particular solution. Examples of the two types are given by

$\mathbf{u'(x)= 24x + x^4 + 3 - \int_0^x (x-t)u(t)dt - \int_0^1 tu(t)dt, u(0) = 0}$ **(I. 2. 38)**

And $\mathbf{u'(x,t)= 1 + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \varepsilon)d\varepsilon d\tau,\ u(0,t) = t^3}$ **(I.2.39)**

II. VOLTERRA THEORY

1 Volterra operator

It was stated in Chapter 1 that Volterra integral equations arise in many scientific applications such as the population dynamics, spread of epidemics, and semi-conductor devices. It was also shown that Volterra integral equations can be derived from initial value problems. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908. Abel considered the problem of determining the equation of a curve in a vertical plane. In this problem, the time taken by a mass point to slide under the influence of gravity along this curve, from a given positive height, to the horizontalaxisisequaltoaprescribedfunctionoftheheight. Abelderivedthe singular Abel's integral equation, a specific kind of Volterra integral equation, that will be studied in a forthcoming chapter. Volterra integral equations, of the first kind or the second kind, are characterized by a variable upper limit of integration. For the first kind Volterra integral equations, the unknown function u(x) occurs only under the integral sign in the form:

$\mathbf{f(x) = \int_0^x k(x,t)\ u(t)\ dt}$ **(II.1)**

However, Volterra integral equations of the second kind, the unknown function u(x) occurs inside and outside the integral sign. The second kind is represented in the form:

$\mathbf{u(x)= f(x) + \lambda \int_0^x k(x,t)u(t)\, dt}$ **(II.2)**

The kernel K(x, t) and the function f(x) are given real-valued functions, and λ is a parameter. A variety of analytic and numerical methods, such as successive approximations method, Laplace transform method, spline collocation method,

Runge-Kuttamethod and others have been used to handle Volterra integral equations. In this text we will apply the recently developed methods, namely, the A domian decomposition method (ADM), the modified decomposition method (mADM), and the variational iteration method (VIM) to handle Volterra integral equations. Some of the traditional methods, namely, successive approximations method, series solution method, and the Laplace transform method will be employed as well. The emphasis in this text will be on the use of these methods and approaches rather than proving theoretical concepts of convergence and existence. The theorems of uniqueness, existence, and convergence are important and can be found in the literature. The concern will be on the determination of the solution u(x) of the Volterra integral equation of first and second kind.

2. Volterra Integral Equations:

2.2.1 Volterra Integral Equations of the Second Kind

We will first study Volterra integral equations of the second kind given by:

$$\mathbf{u(x) = f(x) + \lambda \int_0^x k(x\,,t)u(t)dt} \qquad \textbf{(II.3)}$$

The unknown function u(x), that will be determined, occurs inside and outside the integral sign. The kernel K(x, t) and the function f(x) are given real-valued functions, and λ is a parameter. In what follows we will present the methods, new and traditional, that will be used.

2.2.2 The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian and is well addressed in many references. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral equations as well. The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$\mathbf{u(x) = \sum_{n=0}^{\infty} u_n}\,(x) \qquad \textbf{(II.4)}$$

or equivalently

$$\mathbf{u(x) = u_1(x) + u_1(x) + u_2(x) + \ldots \ldots} \qquad \textbf{(II.5)}$$

Where the components $u_n(x), n > 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components.

$u_0,u_1,u_2,$...individually. As will be seen through the text, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated. To establish the recurrence relation, we substitute (II.5) into the Volterra integral equation (II.4) to obtain

$$\mathbf{\sum_{n=0}^{\infty} u_n\,(x) = f(x) + \lambda \int_0^x k(x\,,t)\left(\sum_{n=0}^{\infty} u_n(t)\right) dt} \textbf{(II.6)}$$

or equivalently

$$\mathbf{u_0(x) + u_1(x) + u_2(x) + f(x) + \lambda \int_0^x k(x\,,t)|u_0(t) + u_1(t) \ldots|dt}$$

$$\boldsymbol{u_0(x) = f(x)}\textbf{(II.7)}$$

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The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x), j > 1$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation:

$$u_{n+1} = \lambda \int_0^x k(x,t)u_n(t)dt, n \geq 0 \qquad \text{(II.8)}$$

that is equivalent to

$$u_0(x) = f(x), \quad u_1(x) = \lambda \int_0^x k(x,t)u_0(t)dt$$

$$u_2(x) = \lambda\int_0^x k(x,t)u_1(t)dt, u_3 = \lambda\int_0^x k(x,t)u_2(t)dt \text{ (II.9)}$$

and so on for other components. In view of (II.9), the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ are completely determined. As a result, the solution $u(x)$ of the Volterra integral equation (II.3) in a series form is readily obtained by using the series assumption in (II.4). It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converges very rapidly to that solution .The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we use the higher accuracy we obtain.

3. Volterra Integro-Differential Equations

Volterra studied the hereditary influences when he was examining a population growth model. The research work resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations was termed as Volterra integro-differential equations , given in the form

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt,$$

Where $u^{(n)}(x) = \frac{d^n u}{dx^n}$

Because the resulted equation in combines the differential operator and the integral operator, then it is necessary to define initial conditions

For the determination of the particular solution $u(x)$ of the Volterra integro-differential equation. Any Volterra integro-differential equation is characterized by the existence of one or more of the derivatives $u(x)$,...outside the integral sign. The Volterra integro-differential equations may be observed when we convert an initial value problem to an integral equation by using Leibnitz rule. The Volterra integro-differential equation appeared after its establishment by Volterra. It then appeared in many physical applications such as glass forming process, Nano hydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. More details about the sources where these equations arise can be found in physics, biology and engineering applications books. To determine a solution for the integro-differential equation, the initial conditions should be given, and this may be clearly seen as a result of involving $u(x)$ and its derivatives. The initial conditions are needed to determine the exact solution.

4. Systems of Volterra Integral Equations

2.4.1 Introduction

Systems of integral equations, linear or nonlinear, appear in scientific applications in engineering, physics, chemistry and populations growth models Studies of systems of integral equations have attracted much concern in applied sciences. The general ideas and the essential features of these systems are of wide applicability. The systems of Volterra integral equations appear in two kinds. For systems of Volterra integral equations of the first kind, the unknown functions appear only under the integral sign in the form:

$$f_1(x) = \int_0^x K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t) + \cdots dt,$$

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$$f_2(x) = \int_0^x K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots dt, \ldots, \quad \text{(II.45)}$$

However, systems of Volterra integral equations of the second kind, the unknown functions appear inside and outside the integral sign of the form:

$$u(x) = f_1(x) + \int_0^x K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \cdots dt,$$

$$v(x) = f_2(x) + \int_0^x K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots dt, \ldots \quad \text{,(II.46)}$$

The kernels $K_i(x,t)$ and$\widetilde{K}_i(x,t)$, and the functions $f_i(x), i = 1, 2, \ldots, n$are given real-valued functions.

A variety of analytical and numerical methods are used to handle systems of Volterra integral equations .The existing techniques encountered some difficulties in terms of the size of computational work, especially when the system involves several integral equations.

To avoid the diffculties that usually arise from the traditional methods, we will use some of the methods presented in this text. The Adomian decomposition method, the Variational iteration method, and the Laplace transform method will form a reasonable basis for studying systems of integral equations. The emphasis in this text will be on the use of these methods rather than proving theoretical concepts of convergence and existence that can be found in other texts.

2.4.2. Systems of Volterra Integral Equations of the Second Kind

We will first study systems of Volterra integral equations of the second kind given by

$$u(x) = f_1(x) + \int_0^x K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \cdots dt,$$

$$v(x) = f_2(x) + \int_0^x K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \cdots dt., \quad \text{(II.47)}$$

The unknown functions $u(x), v(x), \ldots,$ that will be determined appear inside and outside the integral sign. The kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the function $f_i(x)$are given real-valued functions. In what follows we will present the methods, new and traditional, that will be used to handle these systems.

2.4.3. The Adomian Decomposition Method

The Adomian decomposition method was presented before .The method decomposes each solution as an infinite sum of components, where these components are determined recurrently. This method can be used in its standard form, or combined with the noise terms phenomenon. Moreover, the modified decomposition method will be used wherever it is appropriate. It is interesting to point out that the VIM method can also be used, but we need to transform the system of integral equations to a system of integro-differential equations that will be presented later in this chapter.

5. Nonlinear Volterra Integral Equations

2.5.1.Introduction

It is well known that linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908. The linear Volterra integral equations and the linear Volterra integrodifferential equations were presented in previocesection respectively. It is our goal in this chapter to study the nonlinear Volterra integral equations of the first and the second kind. The nonlinear Volterra equations are characterized by at least one variable limit of integration. In the nonlinear

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Volterra integral equations of the second kind, the unknown function u(x) appears inside and outside the integral sign. The nonlinear Volterra integral equation of the second kind is represented by the form

$$\boldsymbol{u(x) = f(x) + \int_0^x K(x,t)F(u(t))dt}\,. \quad \textbf{(II.55)}$$

However, the nonlinear Volterra integral equations of the first kind contains the nonlinear function $F(u(x))$ inside the integral sign.The nonlinear Volterra integral equation of the first kind is expressed in the form

$$\boldsymbol{f(x) = \int_0^x K(x,t)F(u(t))dt}\,. \quad \textbf{(II.56)}$$

For these two kinds of equations, the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and$F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x), sin(u(x))$, and $e^{u(x)}$.

2.5.2.Existence of the Solution for Nonlinear Volterra Integral Equations

In this section we will present an existence theorem for the solution of nonlinear Volterra integral equations. However, in what follows, we present a brief summary of the conditions under which a solution exists for this equation. We first rewrite the nonlinear Volterra integral equation of the second kind by

$$\boldsymbol{u(x) = f(x) + \int_0^x G(x,t,u(t))dt}\,. \quad \textbf{(II.57)}$$

The specific conditions under which a solution exists for the nonlinear Volterra integral equation are:

(i) The function $f(x)$ is integrable and bounded in a

(ii) The function$f(x)$ must satisfy the Lipschitz condition in the interval (a,b). This means that

$$\boldsymbol{|f(x) - f(y)| < k\,|x - y|}. \quad \textbf{(II.58)}$$

(iii) The function $G(x,t,u(t))$ is integrable and bounded $|G(x,t,u(t))| < K$

(iv) The function $G(x,t,u(t))$ must satisfy the Lipschitz condition

$$\boldsymbol{|G(x,t,z) - G(x,t,z'| < M|z - z'|},\textbf{(II.55)}$$

The emphasis in this chapter will be on solving the nonlinear Volterra integral equations rather than proving theoretical concepts of convergence and existence. The theorems of uniqueness, existence, and convergence are important and can be found in the literature. The concern in this text will be on the determination of the solution $u(x)$ of the nonlinear Volterra integral equation of the first and the second kind.

6. Nonlinear Volterra Integral –Differential Equation

2.6.1.Introduction

It is well known that linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. The linear Volterra integro-differential equations were presented. It is our goal in this chapter to study the nonlinear Volterra integrodifferential equations of the first and the second kind. The nonlinear Volterra integro-differential equations are characterized by at least one variable limit of integration. The nonlinear Volterra integro-differential equation of the second kind reads

$$\boldsymbol{u^{(n)}(x) = f(x) + \int_0^x K(x,t)F(u(t))dt} \quad \textbf{(II.60)}$$

And the standard form of the nonlinear Volterra integro-differential equation of the first kind is given by

$$\int_0^x K_1(x,t)F(u(t))dt + \int_0^x K_2(x,t)u^{(n)}(t)dt = f(x), K_2(x,x) \neq 0 \qquad \textbf{(II.61)}$$

Where $u^{(n)}(x)$ is then th derivative of $u(x)$. For these equations, the kernels $K(x,t), K_1(x,t)$ and $K_2(x,t)$, and the function $f(x)$ are given realvalued functions. The function $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x), sin(u(x))$, and $e^{u(x)}$.

2.6.2. Nonlinear Volterra Integro-Differential Equations of the Second Kind

The linear Volterra integro-differential equation, where both differential and integral operators appear together in the same equation, has been studied .

In this section, we will extend the work presented in this section to nonlinear Volterra integro-differential equation. The nonlinear Volterra integro-differential equation of the second kind reads

$$u^n(x) = f(x) + \int_0^x K(x,t)F(u(t))dt \textbf{(II.62)}$$

where $u^{(i)}(x) = \frac{d^i u}{dx^i}$, and $F(u(x))$ is a nonlinear function of $u(x)$.

Because the equation in (II.62) combines the differential operator and the integral operator ,then it is necessary to define initial conditions for the determination of the particular solution $u(x)$ of the nonlinear Volterra integro-differential equation. The nonlinear Volterra integro-differential equation appeared after its establishment by Volterra. It appears in many physical applications such as glass-forming process, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating. More details about the sources where these equations arise can be found in physics, biology and books of engineering applications. We applied many methods to handle the linear Volterra integro-differential equations of the second kind. In this section we will use only some of these methods. However, the other methods presented i can be used as well. In what follows we will apply the combined Laplace transform-Adomian decomposition method, the variational iteration method (VIM), and the series solution method to handle nonlinear Volterra integrodifferential equations of the second kind (II.62).

III. Fredholm theory

1 Introduction

It was stated in Chapter 1 that Fredholm integral equations arise in many scientific applications. It was also shown that Fredholm integral equations can be derived from boundary value problems. Erik Ivar Fredholm (1866– 1927) is best remembered for his work on integral equations and spectral theory. Fredholm was a Swedish mathematician who established the theory of integral equations and his 1903 paper in ActaMathematica played a major role in the establishment of operator theory. As stated before, in Fredholm integral equations, the integral containing the unknown function u(x) is characterized by fixed limits of integration in the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt,$$

where a and b are constants. For the first kind Fredholm integral equations, the unknown function $u(x)$ occurs only under the integral sign in the form

$$f(x) = \int_a^b K(x,t)u(t)dt.$$

However, Fredholm integral equations of the second kind, the unknown function $u(x)$ occurs inside and outside the integral sign. The second kind is represented by the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt.$$

The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions , and λ is a parameter .When

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$f(x) = 0$,the equations saidto be homogeneous. In this chapter, we will mostly use degenerate or separable kernels. A degenerate or a separable kernel is a function that can be expressed as the sum of the product of two functions each depends only on one variable. Such a kernel can be expressed in the form

$$\boldsymbol{K(x,t) = \sum_{i=1}^{n} f_i(x) g_i(t)} \textbf{(III.4)}$$

Examples of separable kernels are $\boldsymbol{x - t}$, $\boldsymbol{(x-t)^2}$, $\boldsymbol{4xt}$, etc. In what follows we state, without proof, the Fredholm alternative theorem

3.1.1 Theorem (Fredholm Alternative Theorem)

If the homogeneous Fredholm integral equation

$$\boldsymbol{u(x) = \lambda \int_a^b K(x,t) u(t) dt} \quad \textbf{(III.5)}$$

Has only the trivial solution $u(x) = 0$, then the corresponding nonhomogeneous Fredholm equation

$$\boldsymbol{u(x) = f(x) + \lambda \int_a^b K(x,t) u(t) dt} \quad \textbf{(III.6)}$$

Always a unique solution. This theorem is known by the Fredholm alternative theorem

3.1.2.Theorem (Unique Solution)

If the kernel $K(x,t)$ in Fredholm integral equation (III.1) is continuous, real valued function, bounded in the square $a \quad x \le b and a \le t \le b, and if f(x)$ is a continuous , real valued function , then a necessary condition for the existence of a unique solution for Fredholm integral equation (III.1) is given by

$$\boldsymbol{|\lambda| M(b-a)} < 1, \qquad \textbf{(III.7)}$$

Where

$$\boldsymbol{|K(x,t)| \le M} \in \mathbb{R} \qquad \textbf{(III.8)}$$

On the contrary, if the necessary condition (III.7) does not hold, then a continuous solution may exist for Fredholm integral equation. To illustrate this, we consider the Fredholm integral equation

$$\boldsymbol{u(x) = -2 - 3x + \int_0^1 (3x+t) u(t) dt} \ \textbf{(III.9)}$$

It is clear that $\lambda = 1$, $|K(x,t)| \le 4$ and $(b-a) = 1$. This gives

$$|\lambda| M(b-a) = 4 \ and (b-a) = 1. \ \textbf{(III.10)}$$

However, the Fredholm equation (III.9) has an exact solution given by

$$\boldsymbol{u(x) = 6x.} \ \textbf{(III.11)}$$

A variety of analytic and numerical methods have been used to handle Fredholm integral equations. The direct computation method, the successive approximations method, and converting Fredholm equation to an equivalent boundary value problem are among many traditional methods that were commonly used. However, in this text we will apply the recently developed methods, namely, the Adomian decomposition method (ADM), the modified decomposition method (ADM), and the variational iteration method (VIM) to handle the Fredholm integral equations. Some of the traditional Methods, namely, successive approximations method, and the direct computation method will be employed as well. The emphasis in this text will be on the use of these methods rather than proving theoretical concepts of convergence and existence. The theorems of uniqueness, existence, and convergence are important and can be found in the literature. The concern will be on the determination of the solution $u(x)$ of the Fredholm integral equations of the first kind and the second kind.

IV. Differential Equation Reducing -Into Volterra and Fredholm Integral Equation and Application

1. Converting IVP to Volterra Integral Equation

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Technology, Management and Research
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In this section, we will study the technique that will convert an initial value problem (IVP) to an equivalent Volterra integral equation and Volterra integro-differential equation as well .For simplicity reasons, we will apply this process to a second order initial value problem given by

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x) \quad \textbf{(IV.1)}$$

Subject to the initial conditions :

$$y(0) = \alpha , y'(0) = \beta , \qquad \textbf{(IV.2)}$$

Where α and β are constants. The functions $p(x)$ and $q(x)$ are analytic functions, and $g(x)$ is continuous through the interval of discussion. To achieve our goal we first set

$$y''(x) = u(x) \quad \textbf{(IV.3)}$$

Where $u(x)$ is a continuous function. Integrating both sides of (IV.3)from 0 to x yields

$$y'(x) - y'(0) = \int_0^x u(t)dt \qquad \textbf{((IV.4)}$$

Or equivalently

$$y'(x) = \beta + \int_0^x u(t)dt \qquad \textbf{(IV.5)}$$

Integrating both sides of (IV.5) from 0 to x yields

$$y(x) - y(0) = \beta x + \int_0^x \int_0^x u(t)\,dtdt , \quad \textbf{(IV.6)}$$

Or equivalently

$$y(x) = \alpha + \beta x + \int_0^x (x-t)u(t)\,dt \quad \textbf{(IV.7)}$$

Obtained upon using the formula that reduce double integral to a single integral that was discussed in the previous section. Substituting (IV.3),(IV.5), and (IV.7) into the initial value problem yields the Volterra integral equation:

$$u(x) + p(x)[\beta + \int_0^x u(t)dt] + q(x)[\alpha + \beta x + \int_0^x (x-t)u(t)dt] = g(x) \quad \textbf{(IV.8)}$$

The last equation can be written in the standard Volterra integral equation form:

$$u(x) = f(x) - \int_0^x K(x,t)u(t)dt \quad \textbf{(IV.9)}$$

$$k(x,t) = p(x) + q(x)(x-t) \quad \textbf{(IV.10)}$$

$$f(x) = g(x) - [\beta p(x) + \alpha q(x) + \beta x q(x) \quad \textbf{(IV.11)}$$

It is interesting to point out that by differentiating Volterra equation (IV.9)

With respect to$x$, using Leibnitz rule, we obtain an equivalent Volterra integro-differential equation in the form:

$$u'(x) + k(x,x)u(x) = f'(x) - \int_0^x \frac{\partial k(x,t)}{\partial x} dt , u(0) = f(0) \quad \textbf{(IV.12)}$$

The technique presented above to convert initial value problems to equivalent Volterra integral equations can be generalized by considering the general initial value problem:

$$y^{(n)} + a_1(x)y^{n-1} + a_{n-1}(x)y' + a_n(x)y = g(x) \quad \textbf{(IV.13)}$$

subject to the initial conditions :

$$y(0) = C_0 , y'(0) = c_1, y''(0) = c_2, \ldots . y^{n-1}(0) = c_{n-1} \textbf{(IV.14)}$$

We assume that the functions are analytic at the origin, and the function g(x) is continuous through the interval of discussion. Let u(x) be a continuous function on the interval of discussion, and we consider the transformation:

$$y^n(x) = u(x) \qquad \textbf{(IV.15)}$$

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Integrating both sides with respect to x gives

$$y^{n-1}(x) = c_{n-1} + \int_0^x u(t)dt \quad \textbf{(IV.16)}$$

Integrating again both sides with respect to x yields

$$y^{n-2}(x) = c_{n-2+} c_{n-1}x \int_0^x \int_0^x u(t)dtdt$$

$$= c_{n-2+} c_{n-1}x + \int_0^x (x-t)\, u(t)dt \quad \textbf{(IV.17)}$$

obtained by reducing the double integral to a single integral. Proceeding as before we find

$$y^{n-3}(x) = c_{n-3} + c_{n-2}x + \frac{1}{2}c_{n-1}x^2 + \int_0^x \int_0^x \int_0^x u(t)dtdtdt$$

$$= c_{n-3} + c_{n-2}x + \frac{1}{2}c_{n-1}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt$$
(1.18)

Continuing the integration process leads to

$$y(x) = \sum_{k=0}^{n-1} \frac{c_k}{k!} x^k + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} u(t)\, dt \quad \textbf{(IV.19)}$$

Substituting (IV.15)– (IV.19) into (IV.13) gives

$$u(x) = f(x) - \int_0^x k(x,t)\, u(t)dt \quad \textbf{(IV.20)}$$ where

$$k(x,t) = \sum_{k=1}^{m} \frac{a_n}{(k-1)!}(x-t)^{k-1} \quad \textbf{(IV.21)}$$ and

$$f(x) = g(x) - \sum_{j=1}^{m} a_j \left(\sum_{k=1}^{j} \frac{c_{n-k}}{(j-k)!} x^{j-k}\right) \textbf{(IV.22)}$$

Notice that the Volterra integro-differential equation can be obtained by differentiating(IV.20) as many times as we like, and by obtaining the initial conditions of each resulting equation.

The following applications will highlight the process to convert initial value problem to an equivalent Volterra integral equation.

V. CONCLUSION

An integral equation is an equation which is one of an indefinite integral. They are important in several physical domains. Maxwell's equations are probably their most famous representatives. They appear in the problems of radiative energy transfer and vibration problems of a rope, a membrane or an axis. The oscillation problems can also be solved using differential equations.
Why we'd want to convert differential equations into integral equations or vis –versa ?

Finding analytical or numerical solutions in the former case is often easier, also qualitative analysis of the asymptotic and singularity behaviour in the phase space.Returning to basics of differential equation, we know that the values of y(x) which satisfy above differential equations are their solutions. Performing a conversion from differential equation in y to integral equation in y is nothing but solving the differential equation for y. After converting an initial value or boundary value problem into an integral equation, we can solve them by shorter methods of integration. This conversion may also be treated as another representation formula for the solution of an ordinary differential equation. A differential equation can be easily converted into an integral equation just by integrating it once or twice or as many times, if needed.But convert a differential equation into Volterra and Fredholm integral equation it request some initial or boundary condition. For that reason in our project which has as title ((the solution of differential equation reduction into Volterra and Fredholm integral equation))

The goal of this project was finding the solution of some differential equation throughout the solution of integral equation.

After seeing some basics notion of differential equation , Volterra and Fredholm integral , we discussed about converting : Converting initial value problem into a Volterra integral equation and Converting boundary value problem into a Fredholm integral equation vis –versa based on some conditions .

We can ask two questions at the end of this work:

1. Can an integral equation always be rewritten as a differential equation ?

2. Can a differential equation always be rewritten as a integral equation?

The question 2 is trivial to answer yes any differential equation can be rewritten as a integral equation but the question 1 which is rewritten any integral equation as a differential , the answer is , In general, no.

An integral equation can be non-local, whereas a differential equation is local (in the sense that it can be described by a function over the jet-bundle). As an illustration. Let $K(x)=\delta_0(x)+\delta_1(x)$ be an integral kernel, where δ_i are the Dirac delta's supported at i. Consider the integral equation, for some fixed smooth $f(x)=\int K(x-y)\phi(y)dy$ for the unknown ϕ. The equation reduces to $\phi(x)+\phi(x+1)=f(x)$. Any continuous function $g(x)$on $[0,1]$ satisfying $g(0)+g(1)=f(0)$ generates a continuous solution of the equation. We challenge you to find a differential equation whose solution set can be thus generated.

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